Dense Point Sets Have Sparse Delaunay Triangulations*

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Delaunay triangulations and Voronoi diagrams are one of the most thoroughly studies objects in computational geometry, with numerous applications including nearest-neighbor searching, clustering, finite-element mesh generation, deformable surface modeling, and surface reconstruction. Many algorithms in these application domains begin by constructing the Delaunay triangulation or Voronoi diagram of a set of points in $\mathbb{R}^3$. Since three-dimensional Delaunay triangulations can have complexity $\Omega(n^2)$ in the worst case, these algorithms have worst-case running time $\Omega(n^2)$. However, this behavior is almost never observed in practice except for highly-contrived inputs. For all practical purposes, three-dimensional Delaunay triangulations appear to have linear complexity.

This frustrating discrepancy between theory and practice motivates our investigation of practical geometric constraints that imply low-complexity Delaunay triangulations. Previous works in this direction have studied *random* point sets under various distributions [7, 6, 13, 11]; *well-spaced* point sets, which are low-discrepancy samples of Lipschitz density functions [4, 15, 16, 17]; and *surface samples* with various density constraints [1, 11].

This paper investigates the complexity of three-dimensional Delaunay triangulations in terms of a global geometric parameter called the *spread*, continuing our work in an earlier paper [11]. The spread of a set of points is the ratio between the largest and smallest interpoint distances. Of particular interest are *dense* point sets in $\mathbb{R}^d$, which have spread $O(n^{1/d})$. Valtr and others [10, 18, 19, 20] have established several combinatorial results for dense point sets that improve corresponding bounds for arbitrary point sets. For other results related to spread, see [3, 5, 12, 14].

Here are our two main results.

Theorem 1. *For any n and Δ, the Delaunay triangulation of any set of n points in $\mathbb{R}^3$ with spread Δ has complexity $O(\Delta^3)$.*

Theorem 2. *For any n and $\Delta \leq n$, there is a set of n points with spread Δ with a regular triangulation of complexity $O(n\Delta)$.*

In particular, the Delaunay triangulation of any dense point set in $\mathbb{R}^3$ has only linear complexity; however, there is a dense set of n points, arbitrarily close to a regular cubical lattice, with a regular triangulation of complexity $\Omega(n^{4/3})$. Theorem 1 is tight in the worst case for all $\Delta = O(\sqrt{n})$ and improves an earlier upper bound of $O(\Delta^4)$ [11]. Theorem 2 was already known for Delaunay triangulations when $\sqrt{n} \leq \Delta \leq n$. A key component of both proofs is the invariance of Delaunay and regular triangulations under certain geometric transformations.

Our proof of Theorem 1 is structured as follows. We implicitly assume that no two points are closer than unit distance apart, so that spread is synonymous with diameter. Two sets P and Q are *well-separated* if each set fits in a ball of radius r, and these two balls are separated by distance $2h$, for some $r \leq h \leq 3r$. Our argument ultimately reduces to counting the number of *crossing edges*—edges in the Delaunay triangulation of $P \cup Q$ with one endpoint in each set. Our proof has four major steps.

- Place a grid of $O(r^2)$ circular *pixels* of constant radius ε on the plane $z = 0$, so that every crossing edge passes through a pixel. Our first step is to prove that the crossing edges intersecting through any pixel all lie within a slab of constant width between two parallel planes. Our proof relies on the fact that the edges of a Delaunay triangulation have a consistent depth order from any viewpoint [8, 9].
- We say that a crossing edge is *relaxed* if its endpoints lie on an empty sphere of radius $O(r)$. We show that at most $O(r)$ relaxed edges pass

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through any pixel, using a generalization of the 'Swiss cheese' packing argument used to prove the earlier $O(\Delta^4)$ upper bound [11]. This implies that there are $O(r^3)$ relaxed crossing edges overall.

- Delaunay triangulations are essentially invariant under *conformal* (*i.e.*, sphere-preserving) transformations. We use this conformal invariance to show that there are a constant number of conformal maps, each changing the spread of $P \cup Q$ by at most a small constant factor, such that every crossing edge of $P \cup Q$ is a relaxed Delaunay edge in at least one image. It follows that $P \cup Q$ has at most $O(r^3)$ crossing edges.

- Finally, we count the Delaunay edges for an arbitrary point set S using an octtree-based well-separated pair decomposition [2]. Every edge in the Delaunay triangulation of S is a crossing edge of some subset pair in the decomposition. However, not every crossing edge is a Delaunay edge; a subset pair contributes a Delaunay edge only if it is close to a large empty *witness* ball. We charge the pair's $O(r^3)$ crossing edges to the $\Omega(r^3)$ volume of this ball. By choosing the witness balls carefully, we ensure that any unit of volume is charged at most a constant number of times, implying the final $O(\Delta^3)$ bound.

In the full paper, we discuss several algorithmic and combinatorial implications of this new upper bound.

Regular triangulations (also called weighted Delaunay triangulations) are orthogonal projections of convex polytopes of one higher dimension [9]. Since affine transformations preserve convexity, it any affine transformation of a regular triangulation is also a regular triangulation. Thus, to prove Theorem 2, it suffices to construct a set S of n points whose Delaunay triangulation has complexity $\Omega(n\Delta)$, such that some affine image of S has spread $O(\Delta)$.

Consider the set of n/Δ line segments $s(i,j)$ with endpoints $(2i, 8j, 0) \pm ((-1)^{i+j}, (-1)^{i+j}, 1)$ for all positive integers $i, j \le \sqrt{n/\Delta}$. Let S be the set of n points containing Δ evenly spaced points on each segment $s(i,j)$. There are $\Omega(\Delta^2)$ Delaunay edges between any pair of adjacent segments $s(i,j)$ and $s(i+1,j)$, and thus the overall complexity of the Delaunay triangulation of S is $\Omega(n\Delta)$. We easily observe that applying the linear transformation $f(x,y,z) = (x, y, \Delta z)$ results in a point set $f(S)$ with spread $O(\Delta)$. This completes the proof of Theorem 2.

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