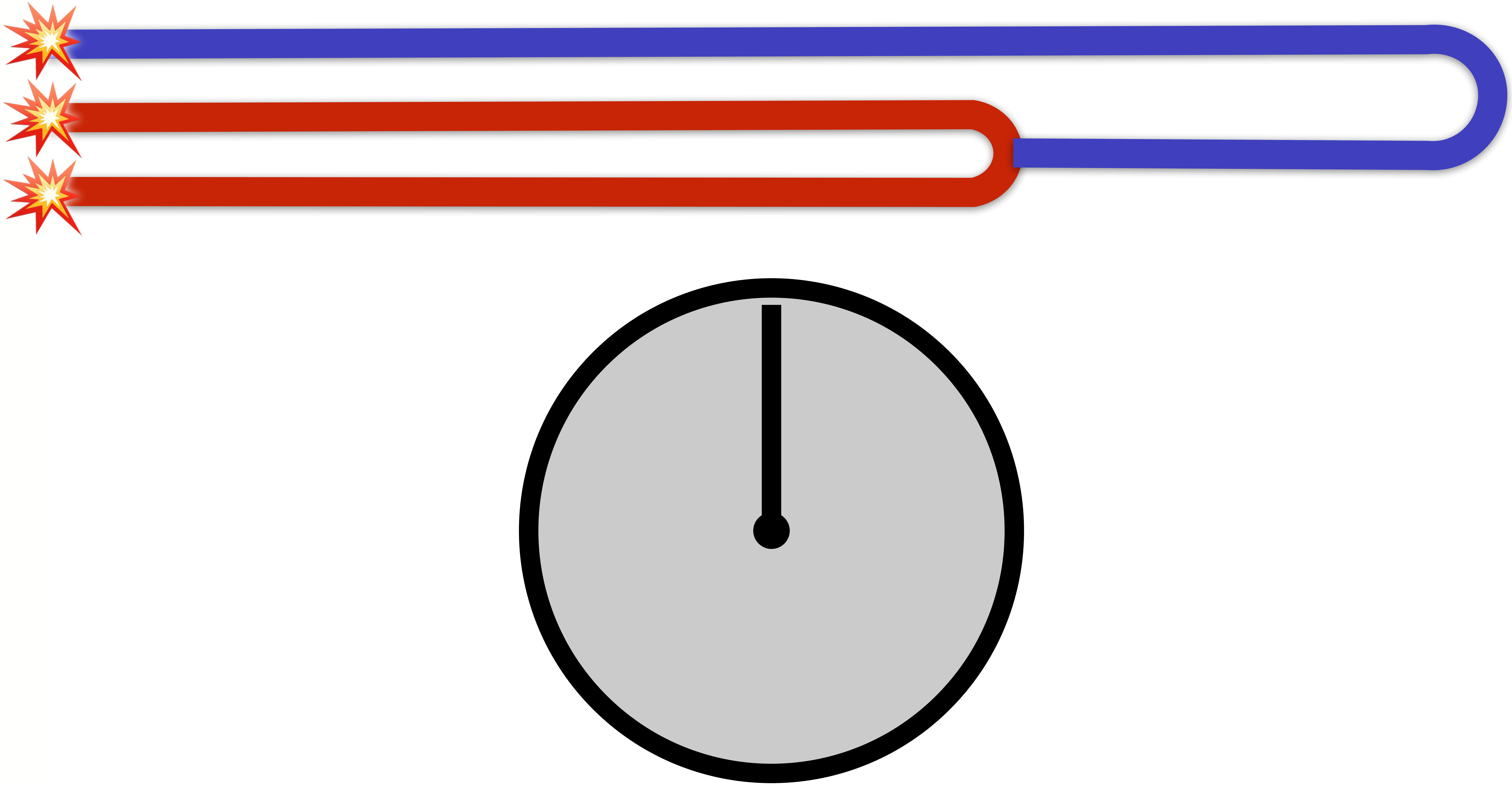
Fusible numbers

Two fuses

A simple puzzle

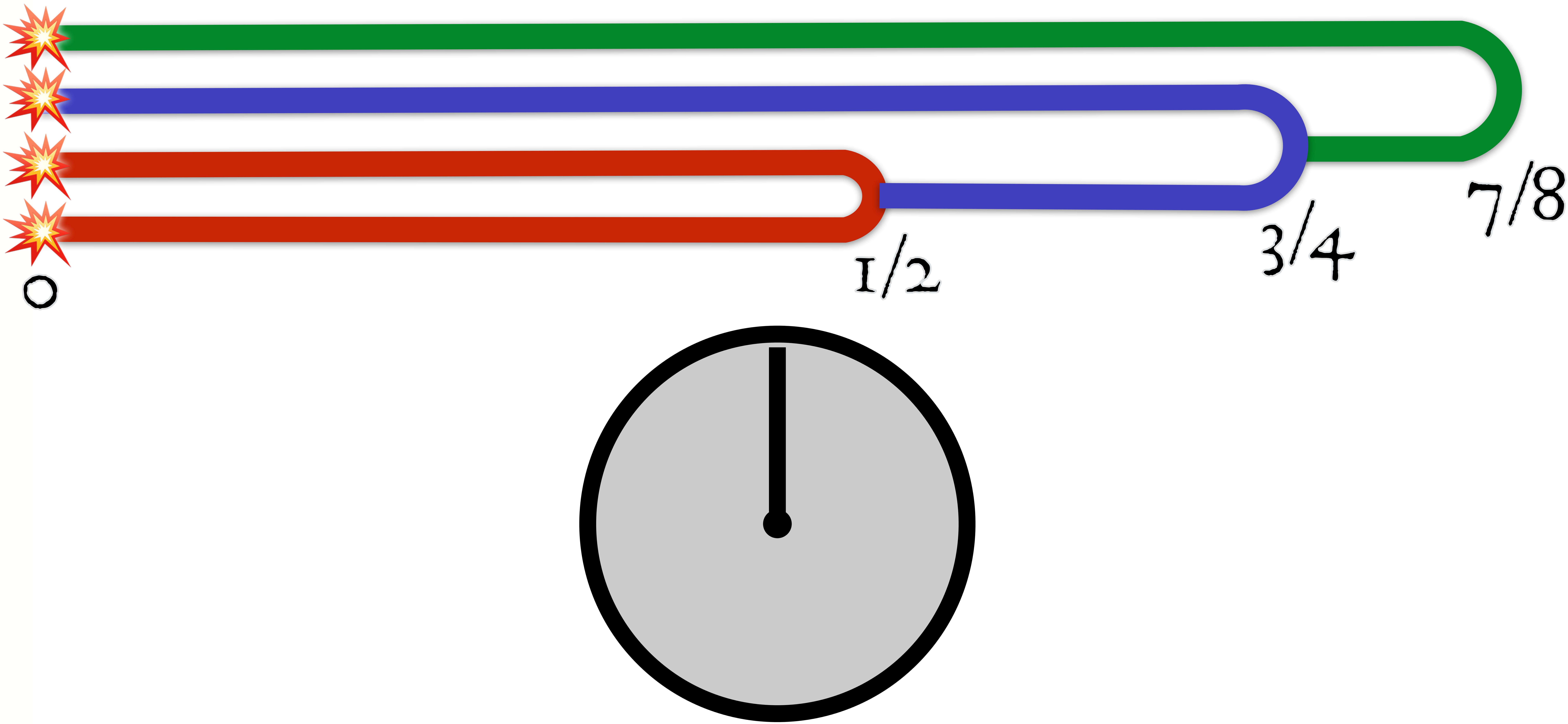
- Suppose you have two fuses (or shoelaces, or pieces of rope, or...).
- Each fuse will burn from one end to the other in exactly one hour, but not necessarily at a fixed rate.
- *How do you accurately measure 45 minutes?*

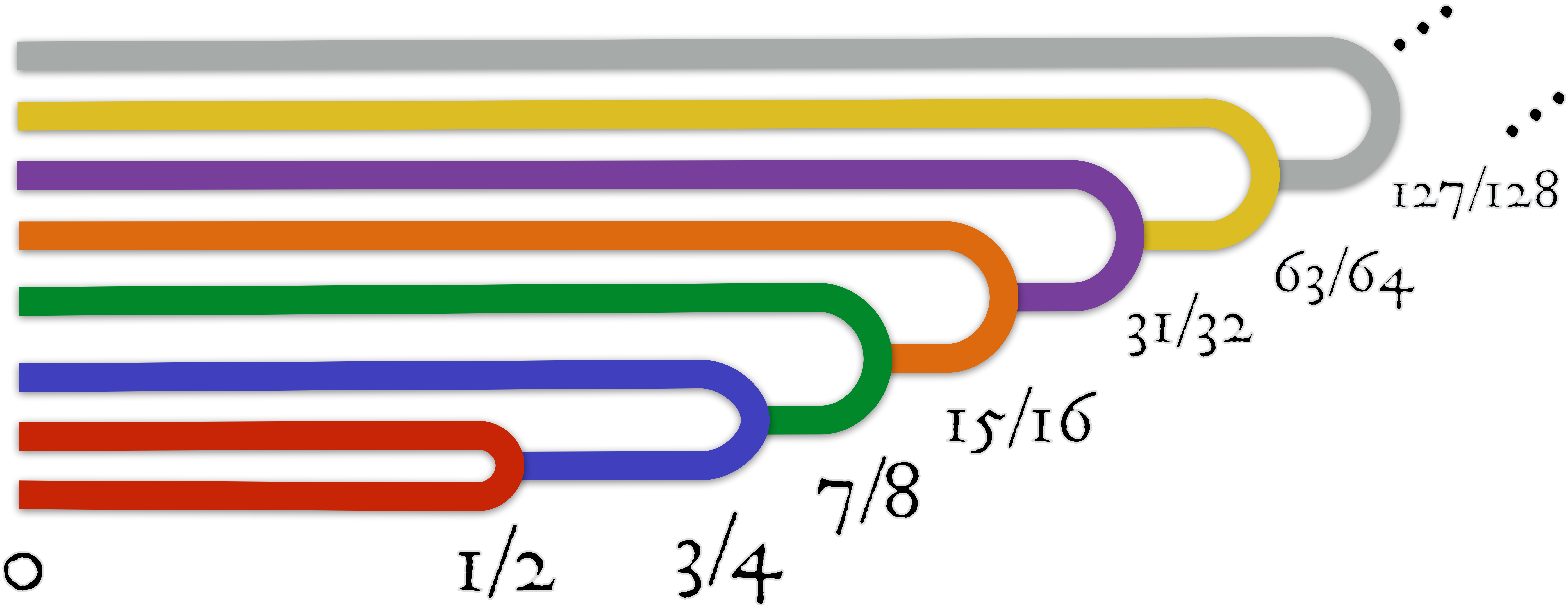


Cool.

But what if we did that more than once?

More fuses!


0
1/2
3/4
7/8



0
1/2
3/4
7/8
15/16
31/32
63/64
127/128
...
...

The Rules

- We can use any *finite* number of fuses.
- We can light any number of fuse ends at the start.
- We can light any number of fuse ends at the exact moment another fuse burns out.
- The timer starts when the first fuse is lit.
- The timer ends when the last fuse burns out.
- *No cheating!* No other clocks, no cutting fuses, no lighting fuses in the middle, no extinguishing fuses, no infinite regress

0 is a ***fusible number***.

If ***x*** and ***y*** are fusible numbers such that $|x-y|<1$, then $\mathbf{(x+y+1)/2}$ is also a fusible number.

These are all the fusible numbers.

$$x \sim y \ = \ (x+y+1)/2$$

"x fuse y"

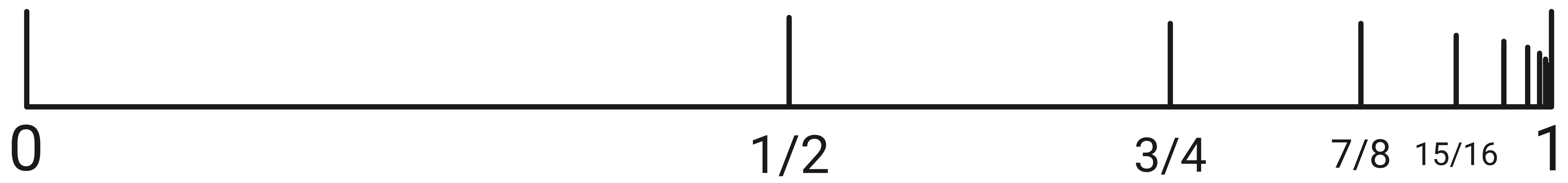
Basic properties

Lemma: The set of fusible numbers is infinite.

Proof: $1 - 2^{-n}$ is fusible for every integer $\mathbf{n} \geq 0$.

$$1 - 2^{-0} = 0 \qquad ✓$$

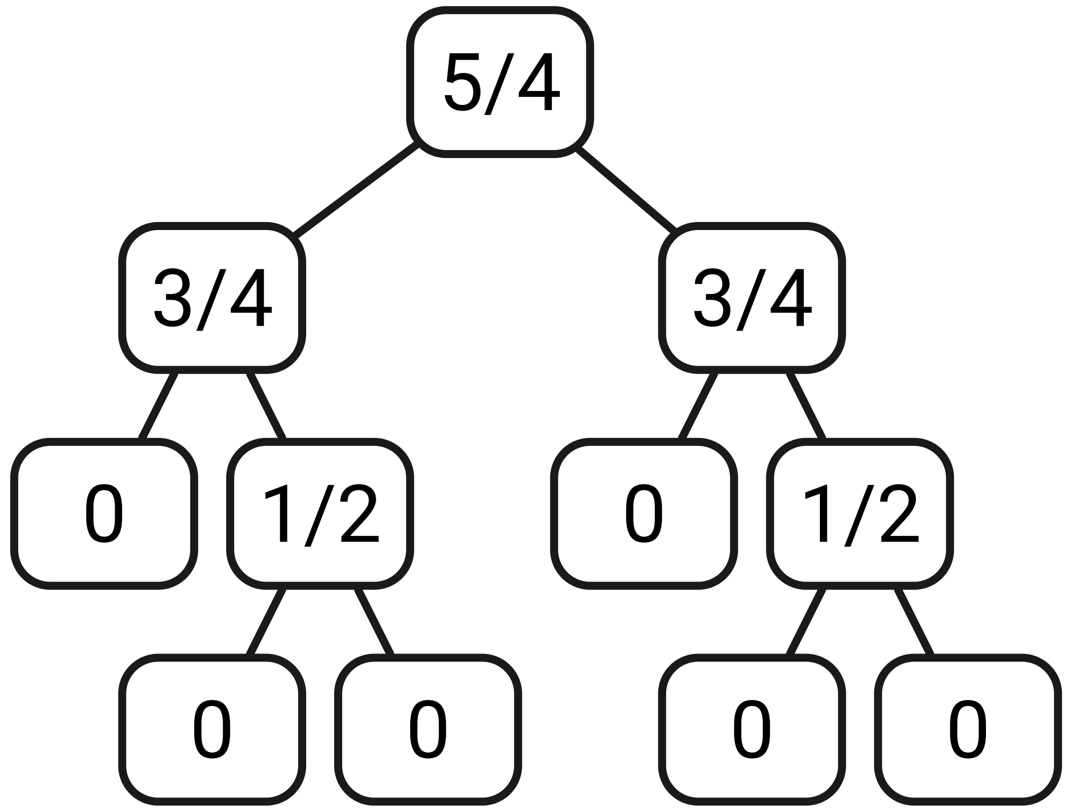
$$1 - 2^{-n} = 0 \sim (1 - 2^{-(n-1)}) \qquad ✓$$

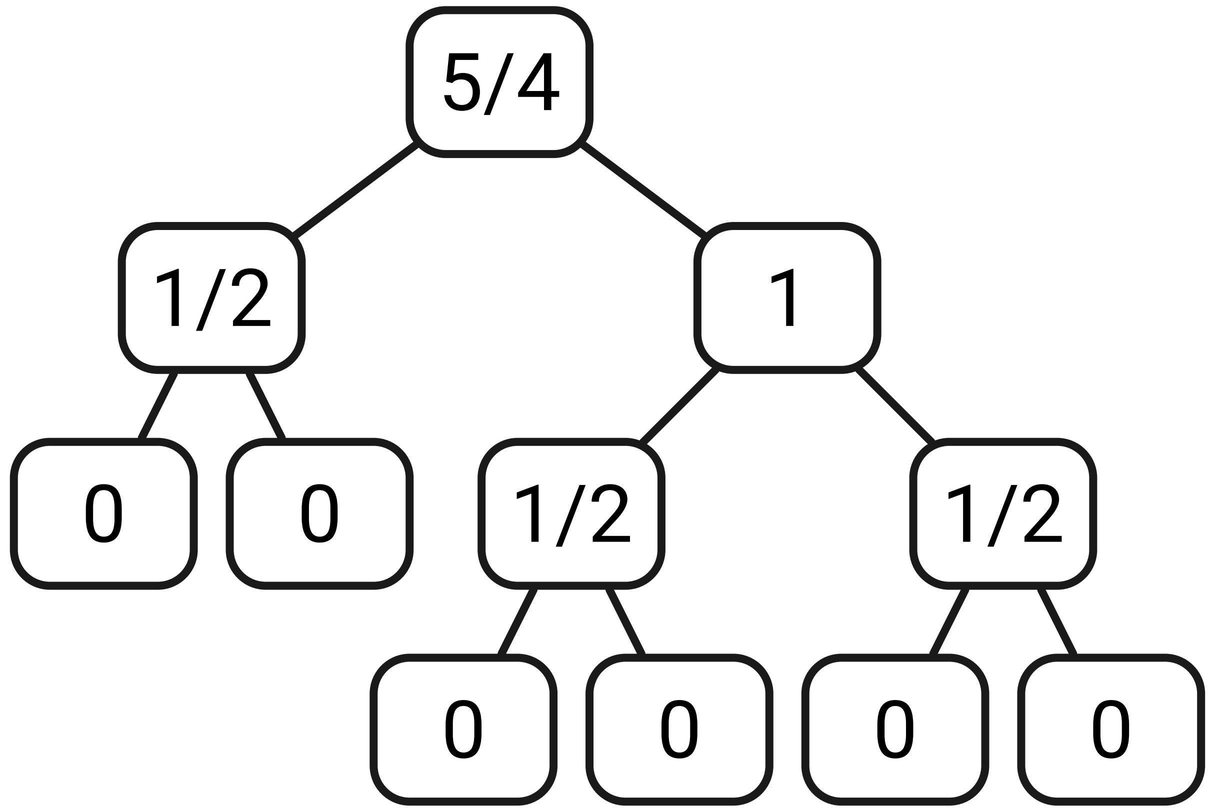


Lemma: The set of fusible numbers is *countable*.

Proof: Every assemblage of fuses can be described by a (not necessarily unique) unordered binary tree.

$$\frac{5}{4} = (0 \sim (0 \sim 0)) \sim (0 \sim (0 \sim 0)) = (0 \sim 0) \sim ((0 \sim 0) \sim (0 \sim 0))$$





$$\text{value}(T) = \frac{1}{2} \sum_{\text{leaf } \ell} 2^{-\text{depth}(\ell)}$$

Theorem: The fusible numbers are ***well-ordered***. Thus, for every real number x, there is a smallest fusible number $> x$.

So we can apply induction on fusible numbers!

Proof:

For the sake of argument, suppose there is an infinite decreasing sequence $x_1 > x_2 > x_3 > \cdots$ of fusible numbers.

This sequence must tend to a limit x.

Without loss of generality, assume x is the *smallest* such limit.

For each index k, we have $x_k = y_k \sim z_k$ for some $y_k \leq z_k$.

Infinite Ramsey theorem $\Rightarrow$ WLOG the infinite sequence $y_1, y_2, y_3, \ldots$ is either decreasing, constant, or increasing.

Suppose $y_1, y_2, y_3, \ldots$ is decreasing. Let $y = \lim y_i$

- Because $y_i \leq x_i - 1/2$ for all i, we have $y < x$, contradicting the minimality of x.

Suppose $y_1, y_2, y_3, \ldots$ is non-decreasing.

- Then $z_1, z_2, z_3, \ldots$ must be decreasing. Let $z = \lim z_i$
- $y_1, y_2, \ldots$ converges to $y = \lim y_i = \lim(2x_i - z_i - 1) = 2x - z - 1$
- Because $z_i \leq x_i$ for all i, we have $z \leq x$ and therefore $z = x$.
- But $y_i \geq y_1 > x_1 - 1 > x - 1$ for all i. So $y > x - 1$ and thus $z > x$.

Small examples

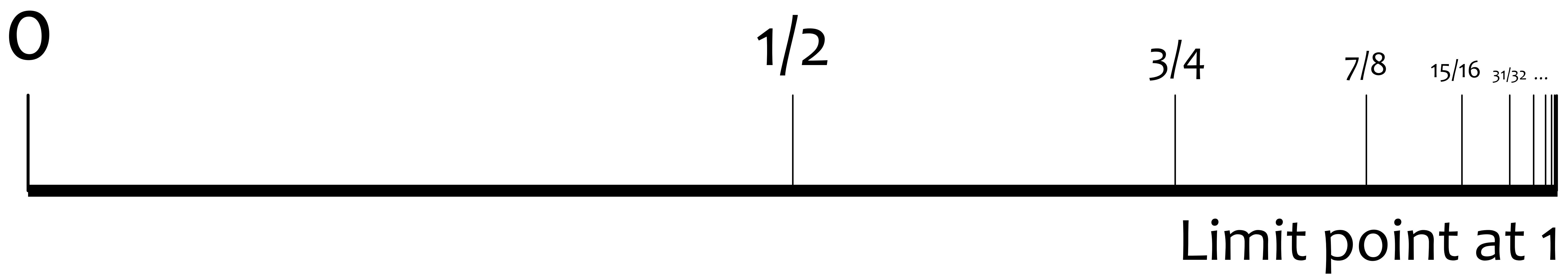
$$0 \sim 0 = 1/2$$

$$0 \sim 1/2 = 3/4$$

$$0 \sim 3/4 = 7/8$$

$$\vdots$$

$$0 \sim (1-2^{-n}) = 1-2^{-(n+1)}$$



Small examples

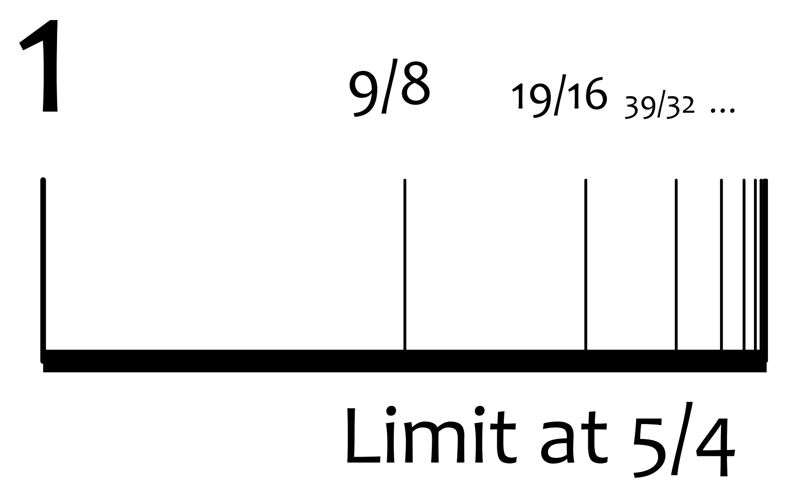
$1/2 \sim 1/2 = 1$

$1/2 \sim 3/4 = 9/8$

$1/2 \sim 7/8 = 19/16$

…

$1/2 \sim (1-2^{-n}) = 5/4-2^{-(n+1)}$



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…

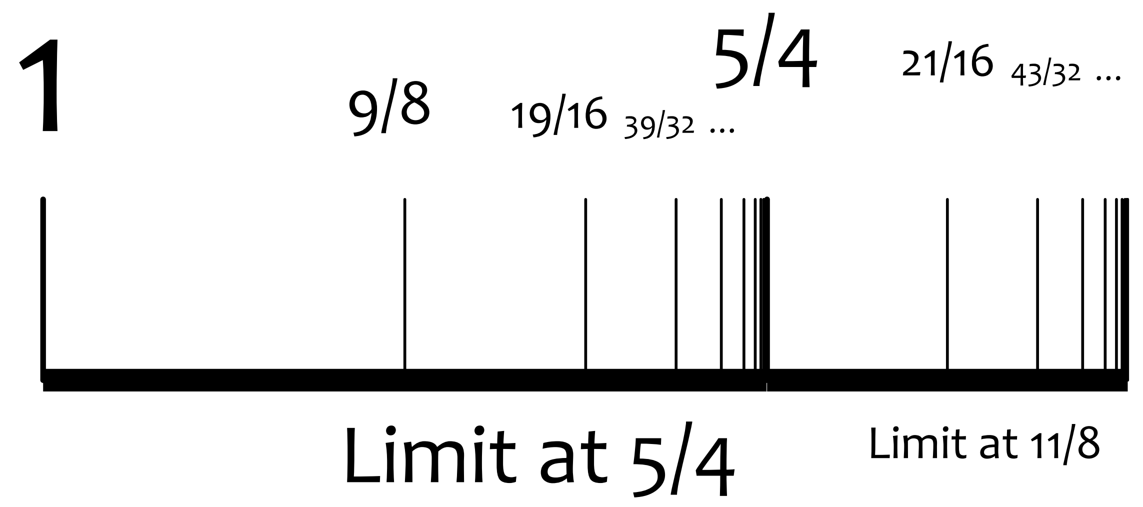
$1/2 \sim (1-2^{-n}) = 5/4-2^{-(n+1)}$

$1/2 \sim 1 = 5/4$

$1/2 \sim 9/8 = 21/16$

…

$1/2 \sim (5/4-2^{-n}) = 11/8-2^{-(n+1)}$



Small examples

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…

$1/2 \sim (1-2^{-n}) = 5/4-2^{-(n+1)}$

$1/2 \sim 1 = 5/4$

$1/2 \sim 9/8 = 21/16$

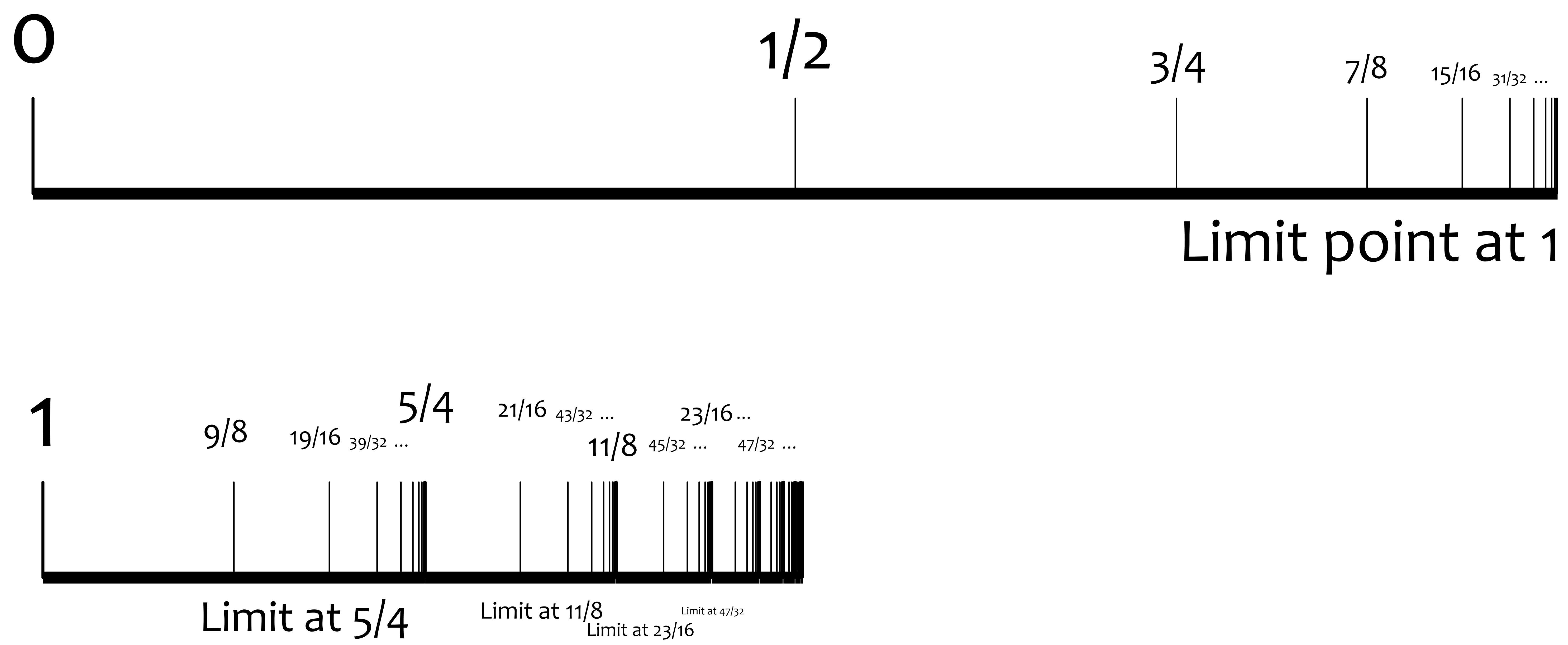
…

$1/2 \sim (5/4-2^{-n}) = 11/8-2^{-(n+1)}$

$1/2 \sim (3/2-2^{-m}-2^{-n}) = 3/2-2^{-m}-2^{-(n+1)}$

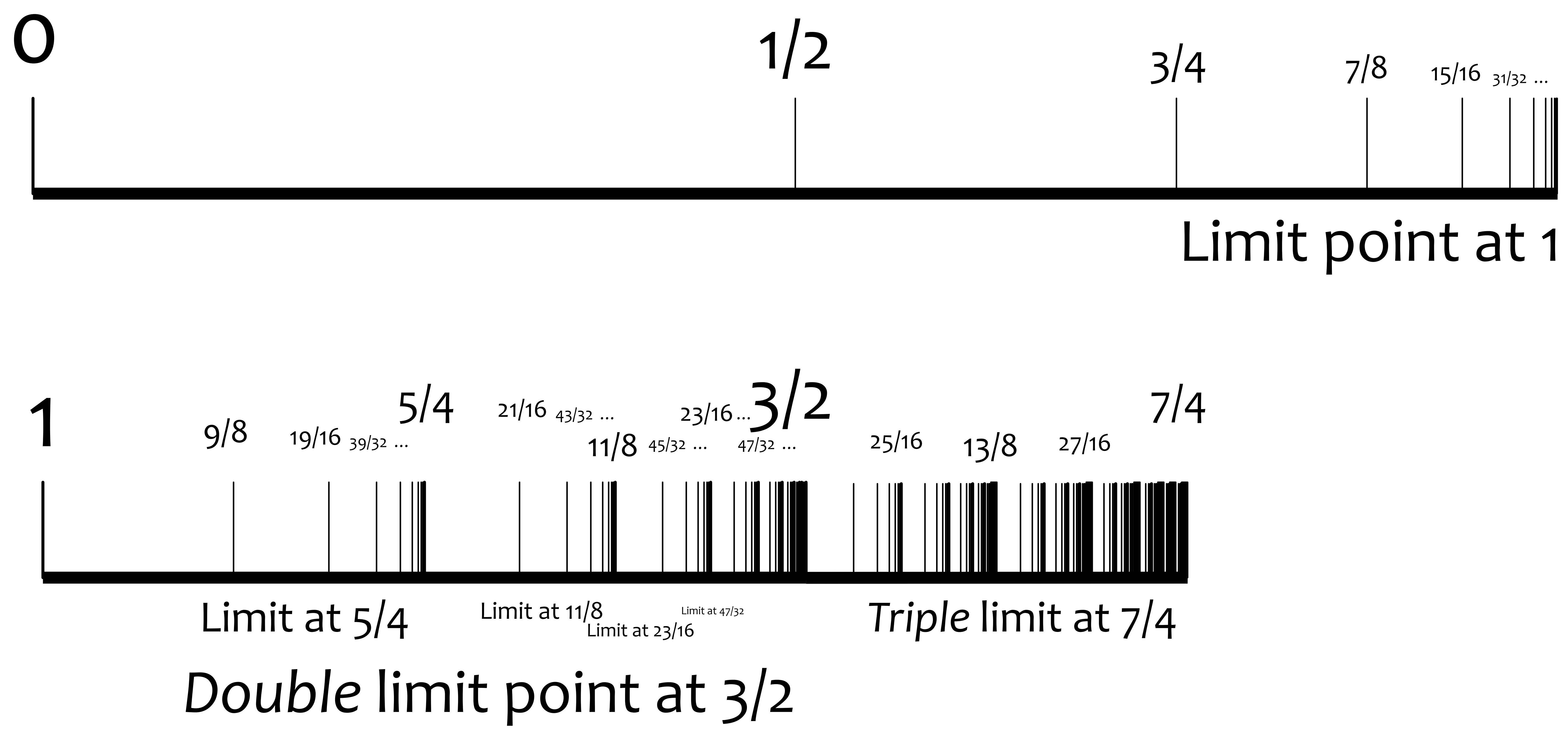
Double limit point at 3/2

Small examples

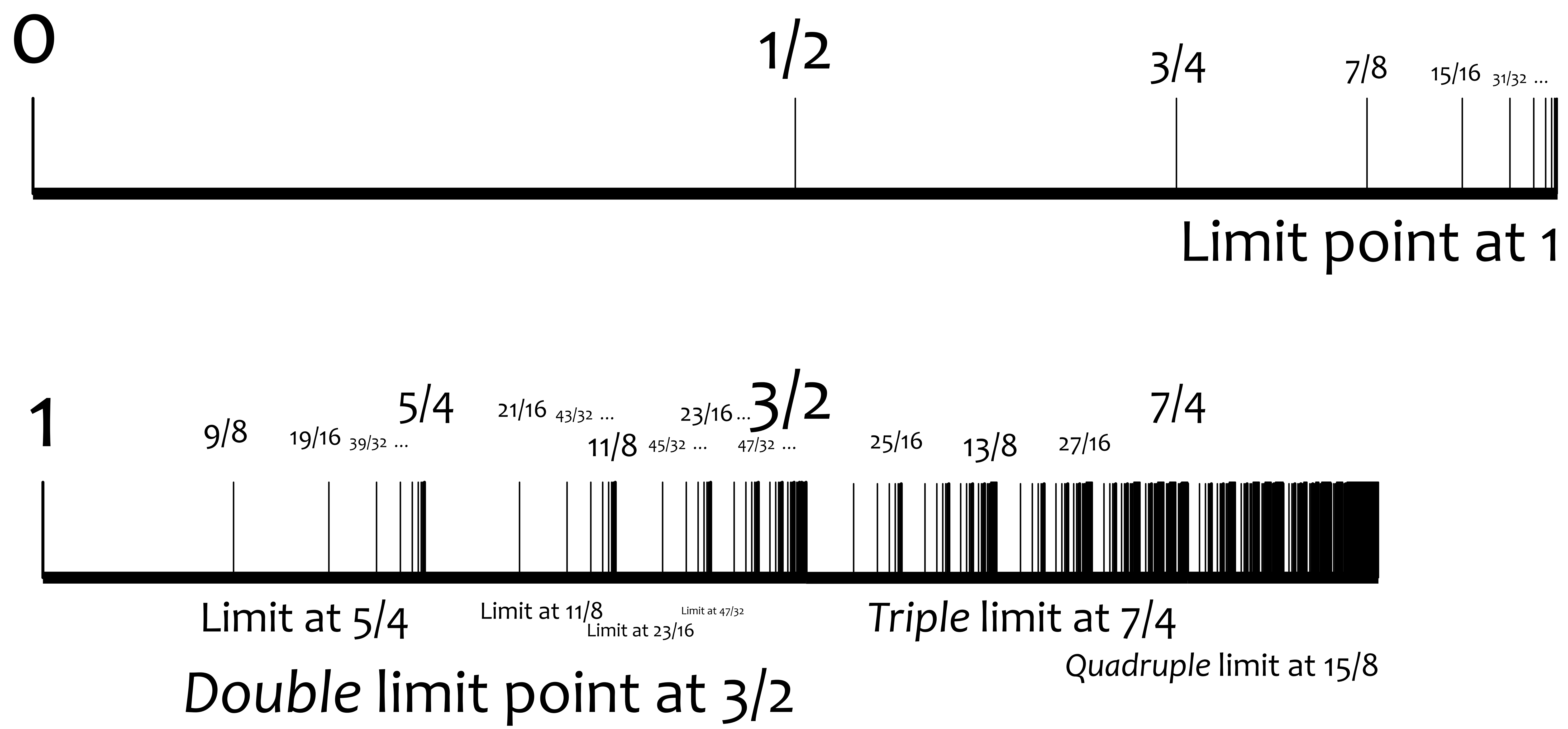


Double limit point at 3/2

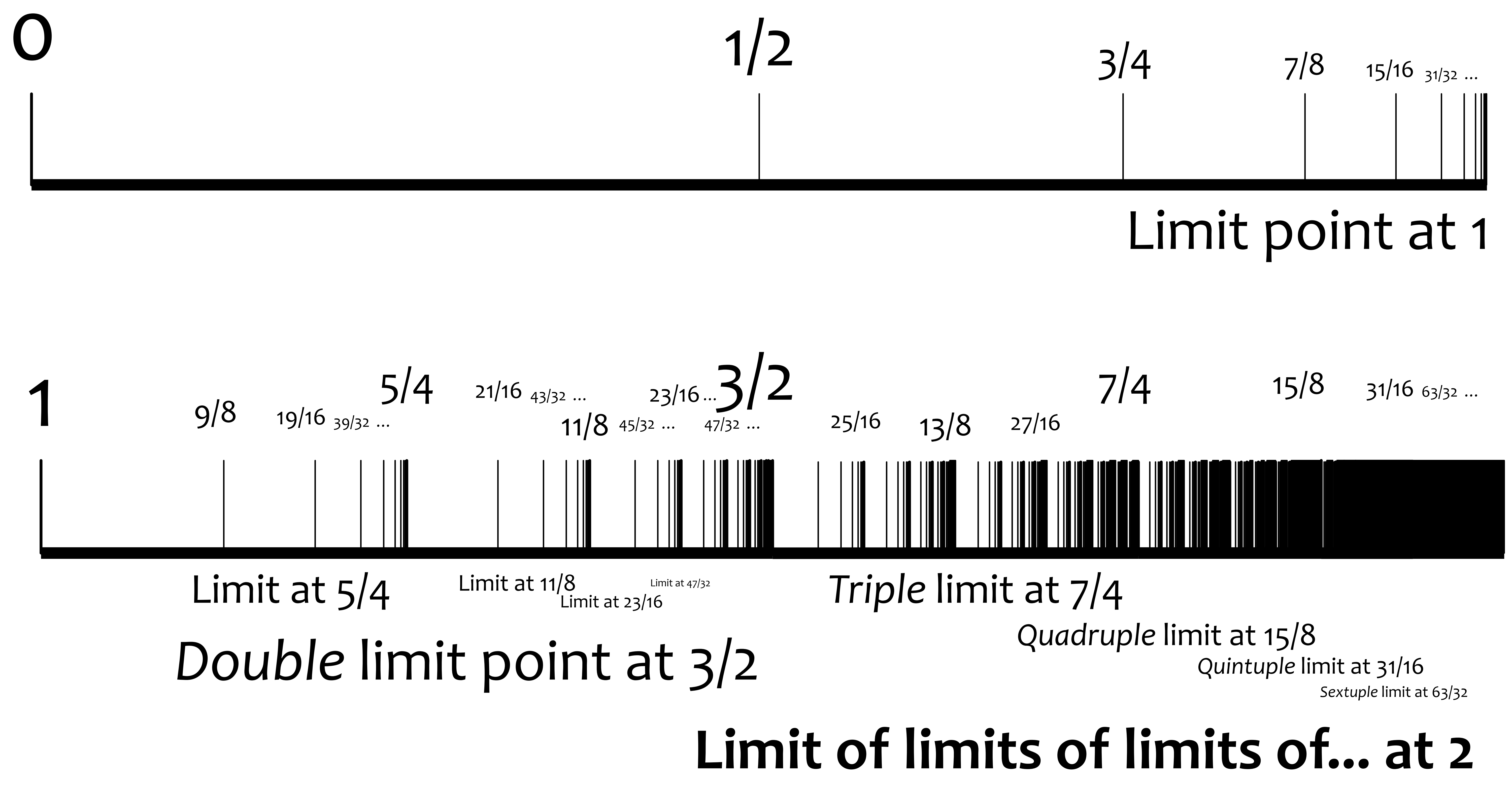
Small examples



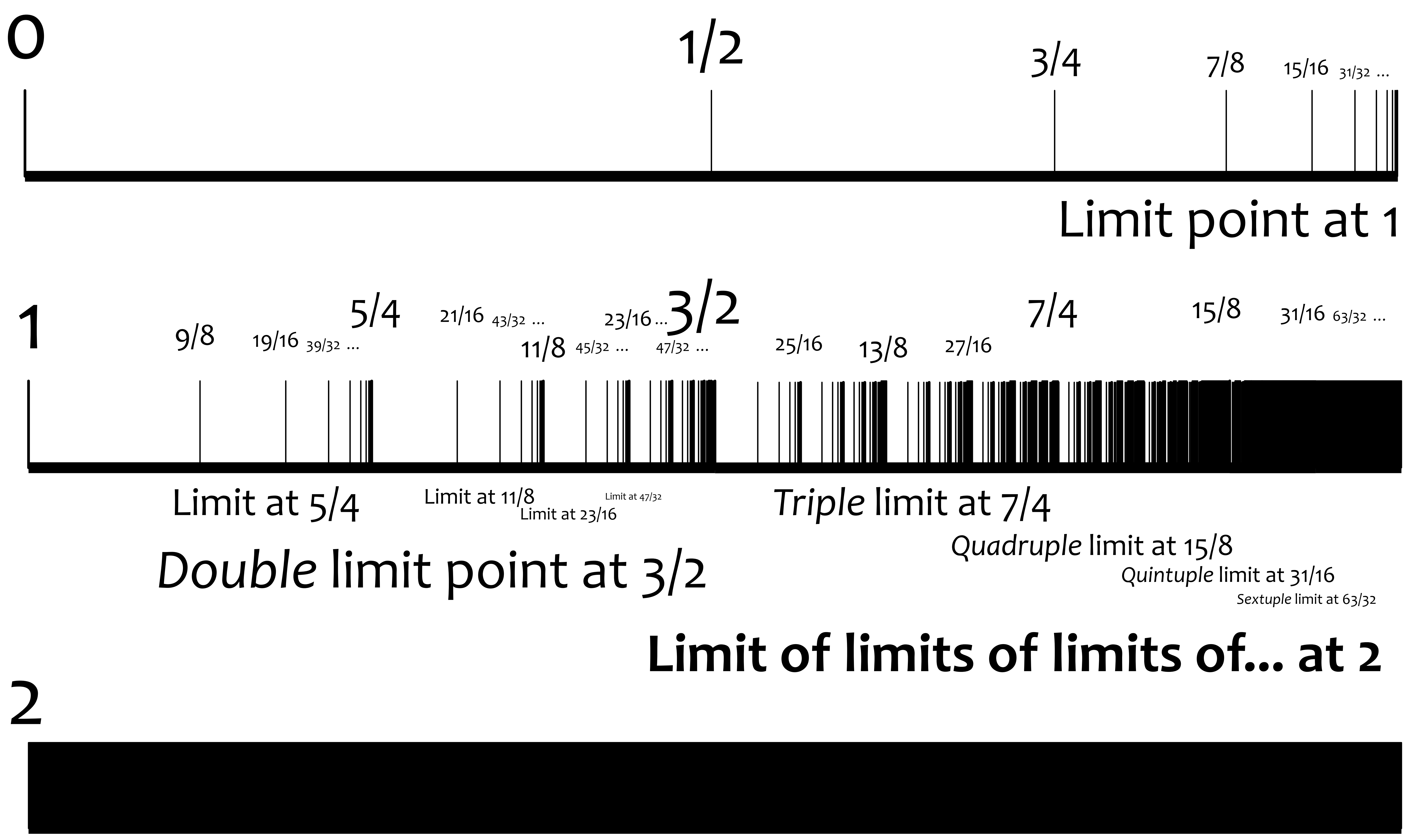
Small examples



Small examples



Small examples



```
# tame successor of x =
# smallest tame fusible > x
def TameSucc(x):
    if x < 0:
        return -x
    y = TameSucc(x-1)
    z = TameSucc(2x-y-1)
    return (y+z-1)/2
```

```
# tame margin of x =
# TameSucc(x) - x
def M(x):
    if x < 0:
        return -x
    return M(x-M(x-1))/2
```

Tame fusible numbers := { TameSucc(x) | x ∈ ℝ }.

Conjecture [E 2010]: Every fusible number is tame.

Xu 2012: Nope! 8449/4096 = 33/16 + 2^{-12} is fusible

but TameSucc(33/16) = 33/16 + 2^{-11}

A simple(!) recurrence

```
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# smallest tame fusible > x
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    z = TameSucc(2x-y-1)
    return (y+z-1)/2
```

```
# tame margin of x =
# TameSucc(x) - x
def M(x):
    if x < 0:
        return -x
    return M(x-M(x-1))/2
```

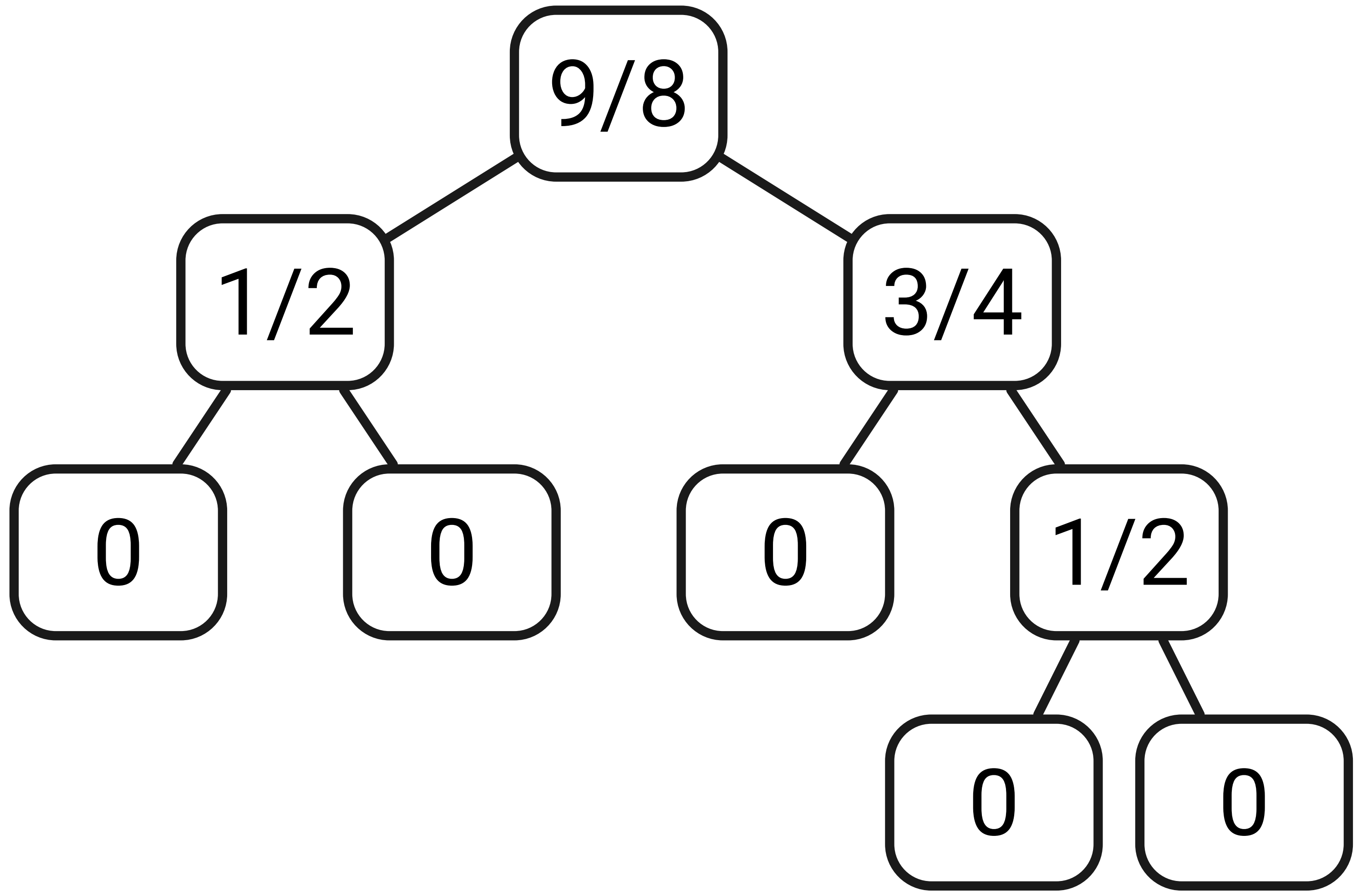
Let *M(x)* = *margin* of *x*

= distance from *x* to the smallest ***tame*** fusible > *x*

$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

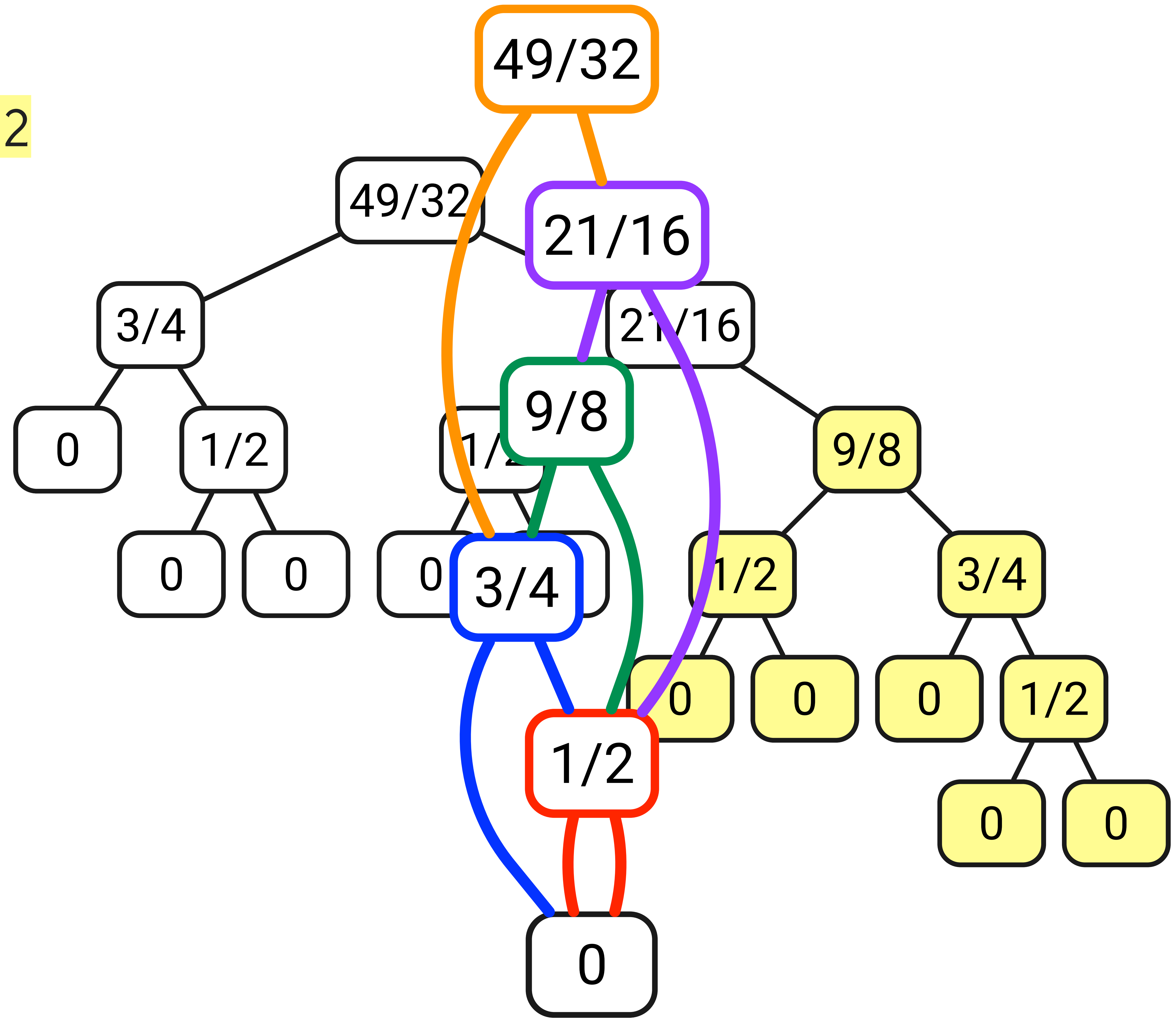
$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

```
M(1) = ···
| M(0) = ···
| | M(-1) = 1
| | M(-1) = 1
| M(0) = 1/2
| M(1/2) = ···
| | M(-1/2) = 1/2
| | M(0) = ···
| | | M(-1) = 1
| | | M(-1) = 1
| | M(0) = 1/2
| M(1/2) = 1/4
M(1) = 1/8
```



$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

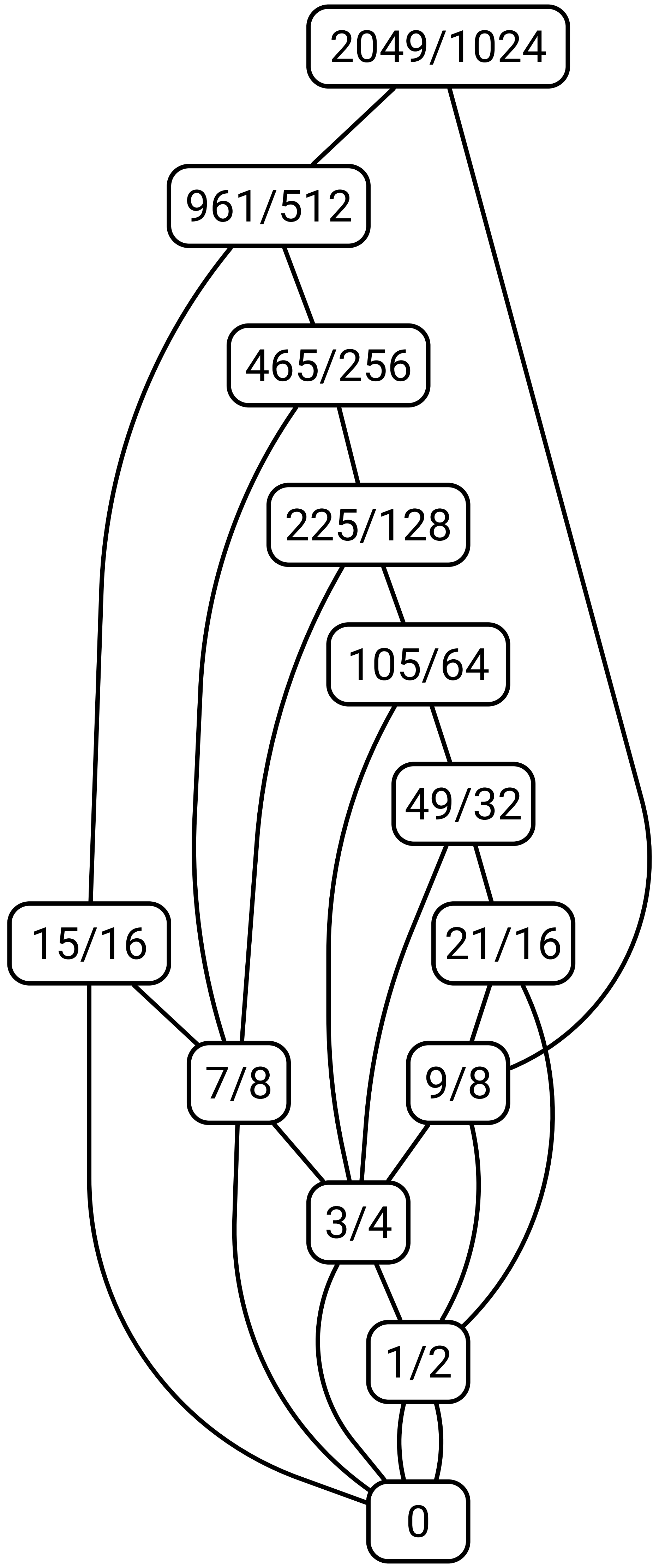
```
M(3/2) = ···
| M(1/2) = ···
| | M(-1/2) = 1/2
| | M(0) = ···
| | | M(-1) = 1
| | | M(-1) = 1
| | M(0) = 1/2
| M(1/2) = 1/4
| M(5/4) = ···
| | M(1/4) = ···
| | | M(-3/4) = 3/4
| | | M(-1/2) = 1/2
| | M(1/4) = 1/4
| | M(1) = ···
| | | M(0) = ···
| | | | M(-1) = 1
| | | | M(-1) = 1
| | | M(0) = 1/2
| | | M(1/2) = ···
| | | | M(-1/2) = 1/2
| | | | M(0) = ···
| | | | | M(-1) = 1
| | | | | M(-1) = 1
| | | | M(0) = 1/2
| | | M(1/2) = 1/4
| | M(1) = 1/8
| M(5/4) = 1/16
M(3/2) = 1/32
```



$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

```
M(2) = ···
| M(1) = ···
| | M(0) = ···
| | | M(-1) = 1
| | | M(-1) = 1
| | M(0) = 1/2
| | M(1/2) = ···
| | | M(-1/2) = 1/2
| | | M(0) = ···
| | | | M(-1) = 1
| | | | M(-1) = 1
| | | M(0) = 1/2
| | M(1/2) = 1/4
| M(1) = 1/8
| M(15/8) = ···
| | M(7/8) = ···
| | | M(-1/8) = 1/8
| | | M(3/4) = ···
| | | | M(-1/4) = 1/4
| | | | M(1/2) = ···
| | | | | M(-1/2) = 1/2
| | | | | M(0) = ···
| | | | | | M(-1) = 1
```

```
| | | | | | M(
| | | | | M(0)
| | | | M(1/2)
| | | M(3/4) =
| | M(7/8) = 1
| | M(29/16) =
| | | M(13/16)
| | | | M(-3/1
| | | | M(5/8)
| | | | | M(-3
| | | | | M(1/
| | | | | | M(
| | | | | | M(
| | | | | M(1/
| | | | M(5/8)
| | | M(13/16)
| | | M(7/4) =
| | | | M(3/4)
| | | | | M(-1
| | | | | M(1/
| | | | | | M(
| | | | | | M(
| | | | | | |
```




```
 = 1
1/2
1/4
3
··
···
 = 3/8
= ···
4) = 3/4
2) = 1/2
= 1/4
1/8
···
= ···
2) = 1/2
= ···
1) = 1
1) = 1
= 1/2
= 1/4
= ···
) = ···
3/4) = 3/4
```

```
| | | | | | | | M(-1/2) = 1/2
| | | | | | | M(1/4) = 1/4
| | | | | | | M(1) = ···
| | | | | | | | M(0) = ···
| | | | | | | | | M(-1) = 1
| | | | | | | | | M(-1) = 1
| | | | | | | | M(0) = 1/2
| | | | | | | | M(1/2) = ···
| | | | | | | | | M(-1/2) = 1/2
| | | | | | | | | M(0) = ···
| | | | | | | | | | M(-1) = 1
| | | | | | | | | | M(-1) = 1
| | | | | | | | | M(0) = 1/2
| | | | | | | | M(1/2) = 1/4
| | | | | | | M(1) = 1/8
| | | | | | M(5/4) = 1/16
| | | | | M(3/2) = 1/32
| | | | M(13/8) = 1/64
| | | M(7/4) = 1/128
| | M(29/16) = 1/256
| M(15/8) = 1/512
M(2) = 1/1024
```

$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

```
| | | | | | | | | | | | | | | | | | | | | | | | | | | M(5/4) = 1/16
| | | | | | | | | | | | | | | | | | | | | | | | | | M(3/2) = 1/32
| | | | | | | | | | | | | | | | | | | | | | | | | M(13/8) = 1/64
| | | | | | | | | | | | | | | | | | | | | | | | M(7/4) = 1/128
| | | | | | | | | | | | | | | | | | | | | | | M(29/16) = 1/256
| | | | | | | | | | | | | | | | | | | | | | M(15/8) = 1/512
| | | | | | | | | | | | | | | | | | | | | M(2) = 1/1024
| | | | | | | | | | | | | | | | | | | | M(33/16) = 1/2048
| | | | | | | | | | | | | | | | | | | M(17/8) = 1/4096
| | | | | | | | | | | | | | | | | | M(69/32) = 1/8192
| | | | | | | | | | | | | | | | | M(35/16) = 1/16384
| | | | | | | | | | | | | | | | M(9/4) = 1/32768
| | | | | | | | | | | | | | | M(73/32) = 1/65536
| | | | | | | | | | | | | | M(37/16) = 1/131072
| | | | | | | | | | | | | M(149/64) = 1/262144
| | | | | | | | | | | | M(75/32) = 1/524288
| | | | | | | | | | | M(19/8) = 1/1048576
| | | | | | | | | | M(153/64) = 1/2097152
| | | | | | | | | M(77/32) = 1/4194304
| | | | | | | | M(309/128) = 1/8388608
| | | | | | | M(155/64) = 1/16777216
| | | | | | M(39/16) = 1/33554432
| | | | | M(313/128) = 1/67108864
| | | | M(157/64) = 1/134217728
| | | M(629/256) = 1/268435456
| | M(315/128) = 1/536870912
| M(79/32) = 1/1073741824
M(5/2) = 1/2147483648
```

$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

Theorem: This recurrence halts for all real inputs.

$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

Proof: Suppose M(x) does not halt but M(z) halts for all $z \leq x-1$.

Let $x_0 = x$ and $x_i = x_{i-1} - M(x_{i-1}-1)$. IH implies this call to M halts.
We have an infinite decreasing sequence $x_0 > x_1 > x_2 > \ldots$.

Let $y_i = x_i-1 = x_{i-1}-1 + M(x_{i-1}-1)$. Then y_i is weak fusible for all $i>0$.

So we have an infinite decreasing sequence $y_1 > y_2 > y_3 > \ldots$ of (weak) fusible numbers, which contradicts *well-ordering*.

Transfinite ordinals

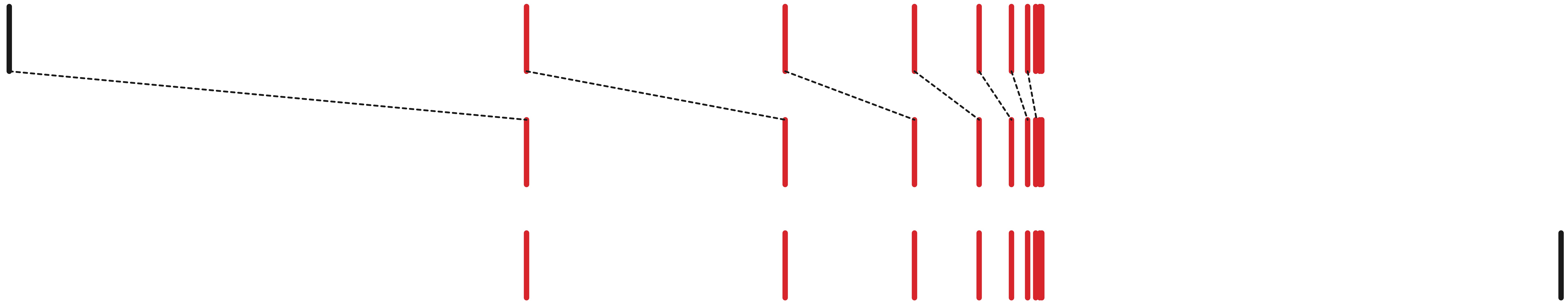
- A ***well-ordered set*** $(X, <)$ is a set X with a total order $<$ such that every non-empty subset of X has a ***smallest*** element with respect to $<$.

- Two well-ordered sets are ***similar*** if there is an order-preserving bijection between them. Equivalence classes are called ***order types*** or ***ordinals***.

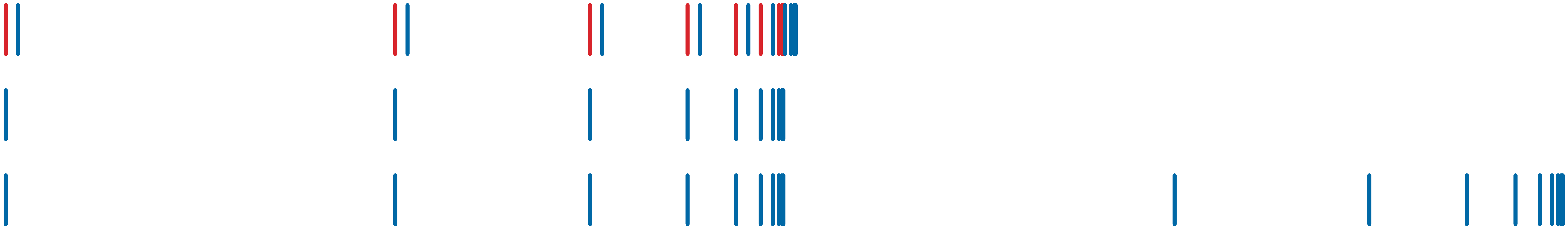
- Finite ***von Neumann*** ordinals, ordered by $< \;=\; \in \;=\; \subset$:
  - $0 = \emptyset$
  - $1 = 0 \cup \{0\} = \{0\} = \{\emptyset\}$
  - $2 = 1 \cup \{1\} = \{0, 1\} = \{ \{\emptyset\}, \{\{\emptyset\}\} \}$
  - $3 = 2 \cup \{2\} = \{0, 1, 2\} = \{ \{\emptyset\}, \{\{\emptyset\}\} , \{\{\emptyset\}, \{\{\emptyset\}\}\} \}$
  - $n = (n-1) \cup \{n-1\} = \{0, 1, 2, \ldots, n-1\}$

- First *transfinite* ordinal: $\omega = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$
  - Then $\omega + 1, \omega + 2, \omega + 3, \ldots, \omega + \omega = \omega\cdot 2$
  - Then $\omega\cdot 2 + 1, \omega\cdot 2 + 2, \omega\cdot 2 + 3, \ldots, \omega\cdot 2 + \omega = \omega\cdot 3$
  - Then $\omega\cdot 3 + 1, \omega\cdot 3 + 2, \ldots, \omega\cdot 4, \omega\cdot 4 + 1, \omega\cdot 4 + 2, \ldots, \omega\cdot 5, \ldots, \omega\cdot 6, \ldots, \omega\cdot\omega = \omega^2$
  - Then $\omega^2 + 1, \omega^2 + 2, \ldots, \omega^2 + \omega, \omega^2 + \omega + 1, \ldots, \omega^2 + \omega\cdot 2, \ldots, \omega^2 + \omega\cdot 3, \ldots, \omega^2\cdot 2, \ldots, \omega^2\cdot 2 + \omega, \ldots, \omega^2\cdot 2 + \omega\cdot 2, \ldots, \omega^2\cdot 2 + \omega\cdot 3, \ldots, \omega^2\cdot 3, \ldots, \omega^2\cdot 4, \ldots, \omega^2\cdot 5, \ldots, \omega^2\cdot\omega = \omega^3$
  - Then $\omega^3 + 1, \ldots\ \omega^3 + \omega, \omega^3 + \omega + 1, \ldots, \omega^3 + \omega\cdot 2, \ldots, \omega^3 + \omega^2, \ldots, \omega^3\cdot 2, \ldots, \omega^4, \ldots, \omega^5, \ldots, \omega^\omega$
  - Then $\omega^\omega + 1, \ldots\ \omega^\omega + \omega, \ldots, \omega^\omega + \omega^2\cdot 2, \ldots, , \omega^{\omega+1}, \ldots, \omega^{\omega\cdot 2}, \ldots, \omega^{\omega^\omega} .., \omega^{\omega^{\omega^\omega}} .., \omega^{\omega^{\omega^{\omega^\omega}}} .., \varepsilon_0$

- Ordinal addition is *not* commutative: $1 + \omega = \omega < \omega + 1$



- Ordinal multiplication is *not* commutative: $2 \cdot \omega = \omega < \omega \cdot 2$



Ord(*x*) = order type of *all* fusibles ≤ *x*

Ord'(*x*) = order type of all *tame* fusibles ≤ *x*

- Ord(0) = 1, Ord(1/2) = 2, Ord(3/4) = 3, Ord(7/8) = 4, ... Ord(1) = ω,
- Ord(9/8) = ω+1, Ord(19/16) = ω+1, Ord(5/4) = ω·2, Ord(3/2) = ω^2
- Ord(7/4) = ω^3. Ord(2) = ω^ω

Ord(*x*) = order type of *all* fusibles ≤ *x*

Ord'(*x*) = order type of all *tame* fusibles ≤ *x*

Theorem: Ord(x) ≥ Ord'(x) for all x.
Ord(x) = Ord'(x) for all x ≤ 2.

Theorem: For every *tame* fusible x, we have $\mathrm{Ord}'(x+1) = \omega^{\mathrm{Ord}'(x)}$.

Corollary: For every integer $n \geq 0$, we have $\mathrm{Ord}'(n) = \omega \uparrow\uparrow n = \underbrace{\omega^{\omega^{\omega^{\cdots^{\omega}}}}}_{n\ \omega\text{'s}}$

Corollary: The order type of the *tame* fusible numbers is ε_0

Corollary: The order type of the fusible numbers is *at least* ε_0

Theorem: For every fusible x, we have $\mathrm{Ord}(x+1) \leq \omega^{\omega^{\mathrm{Ord}(x)}}$

Corollary: For every integer $n \geq 0$, we have $\mathrm{Ord}(n) \leq \omega \uparrow\uparrow 2n = \underbrace{\omega^{\omega^{\omega^{\cdots^{\omega}}}}}_{2n\ \omega\text{'s}}$

Corollary: The order type of the fusible numbers is *at most* ε_0

Corollary: The order type of the fusible numbers is *exactly* ε_0

Fast-growing functions

$$M(x) = \begin{cases} -x & \text{if } x < 0 \\ M(x - M(x-1))/2 & \text{otherwise} \end{cases}$$

Let *g(n)* denote the largest *gap* between arbitrary fusibles ≥ n

n	$-\log_2 M(n)$	$-\log_2 g(n)$
0	1	1
1	3	3
2	10	10
3	*Guess!*	*Guess!*
4	*Guess!*	*Guess!*

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3	1541023937	$> 2 \uparrow^9 16$
4	BIG	**REALLY BIG**

Knuth arrow hierarchy

$$a \uparrow^n b = \begin{cases} a \cdot b & \text{if } n = 0 \\ 1 & \text{if } b = 0 \text{ and } n > 0 \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) & \text{otherwise} \end{cases}$$

Ackermann hierarchy

$$A(0,n) = n + 1$$
$$A(m+1,0) = A(m,1)$$
$$A(m+1,n+1) = A(m, A(m+1,n))$$

This is a good *start*.

Ordinal β is a *successor* ordinal if β has a largest element, or equivalently, if $\beta = \alpha+1$ for some ordinal α. All other ordinals are *limit* ordinals

Every ordinal $\beta < \varepsilon_0$ can be writen in *Cantor normal form*
$\beta = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$ for some ordinals $\beta > \alpha_1 \gg \alpha_2 > \ldots \geq \alpha_k \geq 0$.

Every limit ordinal β is the limit of a *canonical sequence* $\beta[1] < \beta[2] < \beta[3] < \ldots$

- If $\beta = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$ for some k>1, then $\beta[n] = \omega^{\alpha_k}[n]$
- If $\beta = \omega^{\alpha+1} = \omega^{\alpha} \cdot \omega$, then $\beta[n] = \omega^{\alpha} \cdot n$
- If $\beta = \omega^{\alpha}$ for some limit ordinal α, then $\beta[n] = \omega^{\alpha[n]}$.

Intuitively, we get β[n] by *replacing the last ω in the CNF of β with n*.

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Wainer heirarchy

$$F_0(n) = n + 1$$

$$F_{\alpha+1}(n) = F_\alpha^{(n)}(n) \qquad \text{for all } \alpha$$

$$F_\alpha(n) = F_{\alpha[n]}(n) \qquad \text{for all limits } \alpha \leq \varepsilon_0$$

Hardy heirarchy

$$H_0(n) = n$$

$$H_{\alpha+1}(n) = H_\alpha(n+1) \qquad \text{for all } \alpha$$

$$H_\alpha(n) = H_{\alpha[n]}(n) \qquad \text{for all limits } \alpha \leq \varepsilon_0$$

Theorem: $F_\alpha(n) = H_{\omega^\alpha}(n)$ and $F_{\varepsilon_0}(n) = H_{\varepsilon_0}(n)$

Theorem: $-\log_2 g(n) \geq -\log_2 M(n) \geq F_{\varepsilon_0}(n-7)$ for all $n \geq 8$.

n	$-\log_2 M(n)$	$-\log_2 g(n)$
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Peano arithmetic

First order arithmetic: all formulas over

$$\{\forall, \exists, \vee, \wedge, \neg, =, 0, S, +, \cdot\}$$

Peano axioms:

- $\exists x : x = 0 \quad \forall x : S(x) \neq 0 \quad \forall m, n : (S(m) = S(n)) \Rightarrow (m = n)$
- = is reflexive, symmetric, and transitive.
- Recursive definitions of **+** and ·
- Induction scheme: $(\phi(0) \wedge (\forall x : \phi(x) \Rightarrow \phi(S(x)))) \Rightarrow (\forall x : \phi(x))$

Encoding:

"$x > y$" can be encoded as $\exists z : x = y + z + 1$.

"$x \bmod y = z$" can be encoded as $z < y \land \exists q : x = q \cdot y + z$

The ordered pair (x, y) can be encoded as $\binom{x+y}{2} + x$.

Fixed-length tuples can be encoded as nested pairs.

Finite sequences can be encoded using the Chinese Remainder Theorem [Gödel]

Rational numbers, finite sets, trees, graphs

Transfinite ordinals *less than ε_0*

Turing machine behavior for finite time

Encoding:

"x is fusible"

=

"There exists a finite set S of rational numbers that includes x, and such that for every $w \in S$, either $w = 0$ or there exist $y, z \in S$ such that $|z - y| < 1$ and $2w = y+z+1$."

Encoding:

"M(n) terminates for every natural number n"

=

"For all n, there exist m and a finite set S of pairs that contains (n,m) and such that for every $(p,q) \in S$, if $p < 0$ then $q = -p$, and otherwise, there exists q' such that $(p-1, q')$, $(p-q', 2q) \in S$"

Gödel's Incompleteness Theorem:

Every formal system that models arithmetic is either *inconsistent* (contains proofs of some false statements) or *incomplete* (forbids proofs of some true statements).

Peano arithmetic is *obviously* consistent.*

Therefore it must be incomplete.

*equiconsistent with Skolem arithmetic [Gentzen] and several weak models of set theory

Unprovable statements in Peano Arithmetic

ε_0 is a well-ordering [Gentzen]

Goodstein sequences [Kirby Paris]

The Hydra Game [Kirby Paris]

The Worm/Blackboard Game [Hamano and Okada, Beklemishev]

Strengthened finite Ramsey theorem [Paris Harrington]

Variants of Kruskal's tree theorem [Friedman]

Variants of the graph structure theorem [Friedman Robertson Seymour]

Buchholz and Wainer's Theorem:

Let T be a Turing machine that computes a function $g: \mathbb{N} \to \mathbb{N}$;

in particular, T halts on every input.

Suppose Peano Arithmetic can ***prove*** the statement "T halts on every input."

Then for some $\alpha < \varepsilon_0$ and $n_0 \in \mathbb{N}$ we have $g(n) < F_\alpha(n)$ for every $n \geq n_0$.

The function g cannot grow too quickly.

Corollary:

The following true statements are expressible in first-order arithmetic, but not provable in Peano Arithmetic:

“For every integer n, there is a smallest (tame) fusible number > n.”

“The function M(n) halts for every integer n.”

“For every (tame) fusible number x,
there is a maximum-height binary tree whose value is x.”

BOOM goes the dynamite

Any questions?