1. There are (at least) three standard ways to combine polytopes to obtain more complex, higher-dimensional polytopes. Let P be a polytope in $\mathbb{R}^d$, and let Q be a polytope in $\mathbb{R}^e$, where each polytope contains the origin $(0, 0, \ldots, 0)$.

   - The *(Cartesian) product* $P \times Q$ is the following $(d+e)$-dimensional polytope:

   $$P \times Q = \{(p_1, \ldots, p_d,\ q_1, \ldots, q_e) \mid (p_1, \ldots, p_d) \in P,\ (q_1, \ldots, q_e) \in Q\}.$$

     (Convince yourself that this is a polytope!)

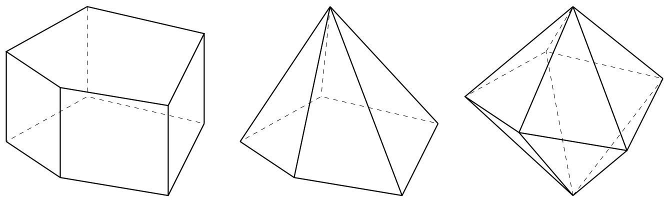
   - The *(orthogonal or direct) sum* $P \oplus Q$ is a $(d+e)$-dimensional polytope obtained by placing P and Q in orthogonal linear subspaces of $\mathbb{R}^{d+e}$ and taking their convex hull. (This definition assumes that P and Q contain the origin.)

   $$\begin{aligned} P \oplus Q = \mathbf{conv}\Big(&\{(p_1, \ldots, p_d, \overbrace{0, \ldots, 0}^{e}) \mid (p_1, \ldots, p_d) \in P\} \cup \\ &\{(\underbrace{0, \ldots, 0}_{d}, q_1, \ldots, q_e) \mid (q_1, \ldots, q_e) \in Q\}\Big) \end{aligned}$$

   - The *join* $P * Q$ is a $(d+e+1)$-dimensional polytope obtained by placing P and Q in two affinely independent (non-intersecting and non-parallel) subspaces of $\mathbb{R}^{d+e+1}$ and taking their convex hull.

   $$\begin{aligned} P * Q = \mathbf{conv}\Big(&\{(p_1, \ldots, p_d, \overbrace{0, \ldots, 0}^{e}, +1) \mid (p_1, \ldots, p_d) \in P\} \cup \\ &\{(\underbrace{0, \ldots, 0}_{d}, q_1, \ldots, q_e, -1) \mid (q_1, \ldots, q_e) \in Q\}\Big) \end{aligned}$$

   For example, a *cylinder* (or *prism*) over a polytope P is the product $P \times I$ with a line segment I. A *pyramid* (or *cone*) over P is the join $P * 0$ with a single point 0 (the *apex*). A *bipyramid* over P is the sum $P \oplus I$ with a line segment I.



A cylinder $P \times I$, a pyramid $P * 0$, and bipramid $P \oplus I$ over a pentagon P.

   We can define some common polytopes by iterating these operations. The d-dimensional hypercube $\square^d$ is the product of d line segments, the d-dimensional cross-polytope $\diamond^d$ is the sum of d line segments, and the d-dimensional simplex $\triangle^d$ is the join of $d+1$ points:

   $$I = 0 * 0 \qquad \square^d = \overbrace{I \times I \times \cdots \times I}^{d}, \qquad \diamond^d = \overbrace{I \oplus I \oplus \cdots \oplus I}^{d}, \qquad \triangle^d = \overbrace{0 * 0 * \cdots * 0}^{d+1}.$$

   Your task is to determine how many faces these polytopes have, as a function of the dimensions and face numbers of their components. We use the standard notation $f_i(X)$ for the number of i-dimensional faces of a polytope X.

Although I've given formal definitions of the various operators and polytopes, I am *not* expecting formal proofs, which would be tedious exercises in linear algebra. A concise, intuitive explanation of your answers is sufficient.

(a) Express $f_i(\triangle^d)$, $f_i(\square^d)$, and $f_i(\Diamond^d)$ as simple functions of i and d. *[Hint: This is trivial if you do the rest of the problem first, but don't.]*

(b) Express $f_i(P \times Q)$ as a function of various $f_j(P)$ and $f_k(Q)$. *[Hint: First consider the cylinder* $P \times I$.*]*

(c) Express $f_i(P \oplus Q)$ as a function of various $f_j(P)$ and $f_k(Q)$. *[Hint: First consider the bipyramid* $P \oplus I$.*]*

(d) Why are the (correct) answers to parts (b) and (c) so similar?

(e) Express $f_i(P * Q)$ as a function of various $f_j(P)$ and $f_k(Q)$. *[Hint: First consider the pyramid* $P * 0$.*]*

2. Let P be a convex polygon in the plane with n vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in counterclockwise order, and let $z_1, z_2, \ldots, z_n$ be arbitrary real numbers. Describe and analyze a randomized algorithm to construct the convex hull of the sequence of points (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$ in $O(n)$ expected time.

3. (a) Given a set of n points in general position in the plane, *exactly* how many vertices does its Voronoi diagram have? *[Hint: The answer depends on another parameter of the point set.]*

   (b) Describe an algorithm to reconstruct a set of n points from its Voronoi diagram in $O(n)$ time. The input is the Voronoi graph—the x- and y-coordinates of the Voronoi vertices (including the point at infinity), the line equations of the Voronoi edges, and the abstract graph structure as an adjacency list. What does your algorithm do if the input is not actually a Voronoi diagram? *[Hint: First solve the special cases* $n = 3$ *and* $n = 4$.*]*

   *(c) Now suppose we want to reconstruct a set of n points given only the v *vertices* of its Voronoi diagram and the integer n. In general, this is impossible. If the number of Voronoi vertices is less than the number of original points, we simply do not have enough information to specify the original points; we have $2n$ degrees of freedom, but only $2v < 2n$ constraints.

   Is this the only contraint? Can we reconstruct any set of n points from its v Voronoi vertices if $v \geq n$? Either describe and analyze an algorithm (don't worry about making it efficient), or describe an ambiguous set of Voronoi vertices. If necessary, assume the *output* points are in general position.

Finally, our proofs are (meant to be) easy and transparent, following simple geometric ideas through some elementary linear algebra, so they might even be fun. (There should be no crying in this lecture.)

— Günter M. Ziegler, *Lectures on Polytopes* (1995)