1. Probability

   (a) n people have checked their hats with a hat clerk. The clerk is somewhat absent-minded and returns the hats uniformly at random (with no regard for whether each hat is returned to its owner). On average, how many people will get back their own hats?

   (b) Let S be a uniformly random permutation of $\{1, 2, \ldots, n-1, n\}$. As we move from the left to the right of the permutation, let X denote the smallest number seen so far. On average, how many different values will X take?

2. A *tournament* is a directed graph where each pair of distinct vertices u, v has either the edge uv or the edge vu (but not both). A *Hamiltonian path* is a (directed) path that visits each vertex of the (di)graph. Prove that every tournament has a Hamiltonian path.

3. Describe and analyze a data structure that stores a set of n records, each with a numerical *key*, such that the following operation can be performed quickly:

   `Foo`(a): return the sum of the records with keys at least as large as a.

   For example, if the keys are:

   $$3\ 4\ 9\ 6\ 5\ 8\ 7\ 1\ 0$$

   then `Foo`(2) would return 42, since $3, 4, 5, 6, 7, 8, 9$ are all larger than 2 and $3+4+5+6+7+8+9 = 42$.

   You may assume that no two records have equal keys, and that no record has a key equal to a. Analyze both the size of your data structure and the running time of your `Foo` algorithm. Your data structure must be as small as possible and your `Foo` algorithm must be as fast as possible.