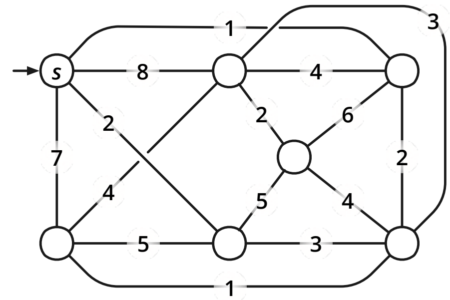
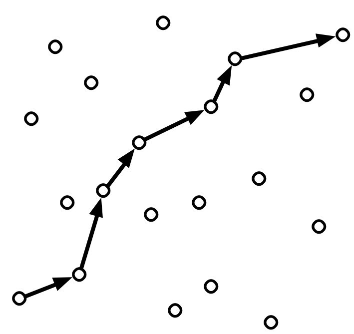
This exam lasts 120 minutes.
Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. ***Clearly*** indicate the following spanning trees in the weighted graph pictured below. Some of these subproblems have more than one correct answer.

   (a) A depth-first spanning tree rooted at s
   (b) A breadth-first spanning tree rooted at s
   (c) A shortest-path tree rooted at s
   (d) A minimum spanning tree
   (e) A *maximum* spanning tree



2. A ***polygonal path*** is a sequence of line segments joined end-to-end; the endpoints of these line segments are called the ***vertices*** of the path. The ***length*** of a polygonal path is the sum of the lengths of its segments. A polygonal path with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ is ***monotonically increasing*** if $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for every index i—informally, each vertex of the path is above and to the right of its predecessor.



A monotonically increasing polygonal path with seven vertices through a set of points

Suppose you are given a set S of n points in the plane, represented as two arrays $X[1..n]$ and $Y[1..n]$. Describe and analyze an algorithm to compute the length of the maximum-length monotonically increasing path with vertices in S. Assume you have a subroutine $\textsc{Length}(x, y, x', y')$ that returns the length of the segment from (x, y) to (x', y').

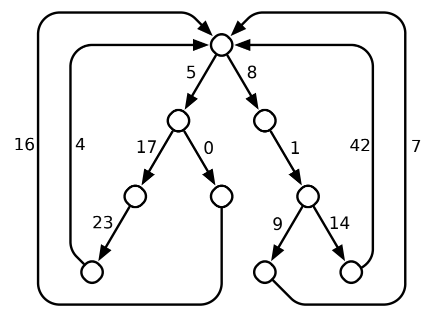
3. Suppose you are maintaining a circular array $X[0..n-1]$ of counters, each taking a value from the set $\{0,1,2\}$. The following algorithm increments one of the counters; if the counter overflows, the algorithm resets it 0 and recursively increments its two neighbors.

```
INCREMENT(i):
  X[i] ← X[i] + 1
  if X[i] = 3
      X[i] ← 0
      INCREMENT((i − 1) mod n)
      INCREMENT((i + 1) mod n)
```

(a) Suppose $n = 5$ and $X = [2,2,2,2,2]$. What does X contain after we call INCREMENT(3)?

(b) Suppose all counters are initially 0. ***Prove*** that INCREMENT runs in $O(1)$ amortized time.

4. A *looped tree* is a weighted, directed graph built from a binary tree by adding an edge from every leaf back to the root. Every edge has non-negative weight.



A looped tree.

(a) How much time would Dijkstra's algorithm require to compute the shortest path from an arbitrary vertex s to another arbitrary vertex t, in a looped tree with n vertices?

(b) Describe and analyze a faster algorithm. Your algorithm should compute the actual shortest path, not just its length.

5. Consider the following algorithm for finding the smallest element in an unsorted array:

```
RANDOMMIN(A[1..n]):
  min ← ∞
  for i ← 1 to n in random order
      if A[i] < min
          min ← A[i]    (⋆)
  return min
```

Assume the elements of A are all distinct.

(a) In the worst case, how many times does RANDOMMIN execute line $(\star)$?

(b) What is the probability that line $(\star)$ is executed during the *last* iteration of the for loop?

(c) What is the *exact* expected number of executions of line $(\star)$?