Consider the following recursively defined function on strings:

$$stutter(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet stutter(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

Intuitively, $stutter(w)$ doubles every symbol in w. For example:

- $stutter(\texttt{PRESTO}) = \texttt{PPRREESSTTOO}$
- $stutter(\texttt{HOCUS}\diamond\texttt{POCUS}) = \texttt{HHOOCCUUSS}\diamond\diamond\texttt{PPOOCCUUSS}$

Let L be an arbitrary regular language.

1. Prove that the language $stutter^{-1}(L) := \{w \mid stutter(w) \in L\}$ is regular.
2. Prove that the language $stutter(L) := \{stutter(w) \mid w \in L\}$ is regular.

Work on these later:

3. Let L be an arbitrary regular language.

   (a) Prove that the language $insert1(L) := \{x1y \mid xy \in L\}$ is regular.
   Intuitively, $insert1(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one 1. For example, if $L = \{\varepsilon, \texttt{OOK!}\}$, then $insert1(L) = \{\texttt{1}, \texttt{1OOK!}, \texttt{O1OK!}, \texttt{OO1K!}, \texttt{OOK1!}, \texttt{OOK!1}\}$.

   (b) Prove that the language $delete1(L) := \{xy \mid x1y \in L\}$ is regular.
   Intuitively, $delete1(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one 1. For example, if $L = \{\texttt{101101}, \texttt{00}, \varepsilon\}$, then $delete1(L) = \{\texttt{01101}, \texttt{10101}, \texttt{10110}\}$.

4. Consider the following recursively defined function on strings:

$$evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot evens(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

   Intuitively, $evens(w)$ skips over every other symbol in w. For example:

   - $evens(\texttt{EXPELLIARMUS}) = \texttt{XELAMS}$
   - $evens(\texttt{AVADA}\diamond\texttt{KEDAVRA}) = \texttt{VD}\diamond\texttt{EAR}$.

   Once again, let L be an arbitrary regular language.

   (a) Prove that the language $evens^{-1}(L) := \{w \mid evens(w) \in L\}$ is regular.

   (b) Prove that the language $evens(L) := \{evens(w) \mid w \in L\}$ is regular.