CS/ECE 374 A ✧ Fall 2023

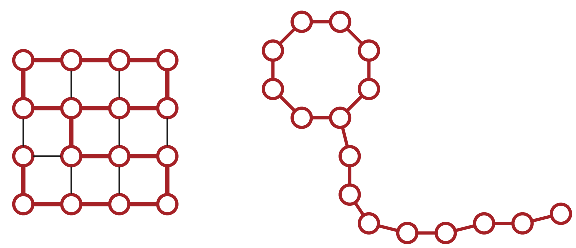
Homework 11

Due Tuesday, November 28, 2023 at 9pm

This is the last graded homework before the final exam.

---

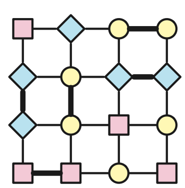
1. A *balloon* of size ℓ is an undirected graph consisting of a (simple) cycle of length ℓ and a (simple) path of length ℓ, where one endpoint of the path lies on the cycle, and otherwise the cycle and the path are disjoint. Every balloon of size ℓ has exactly 2ℓ vertices and 2ℓ edges. For example, the 4×4 grid graph shown below contains a balloon subgraph of size 8.



Prove that it is NP-hard to find the size of the the largest balloon subgraph of a given undirected graph.

2. Recall that a *3-coloring* of a graph assigns each vertex one of three colors, say red, yellow, and blue. A 3-coloring is *proper* if every edge has endpoints with different colors. The 3Color problem asks, given an arbitrary undirected graph G, whether G has a proper 3-coloring.

Call a 3-coloring of a graph G *slightly improper* if each vertex has *at most one neighbor* with the same color. The SlightlyImproper3Color problem asks, given an arbitrary undirected graph G, whether G has a slightly improper 3-coloring.



(a) Consider the following attempt to prove that SlightlyImproper3Color is NP-hard, using a reduction from 3Color.

Non-solution: We reduce from 3Color. Given an arbitrary input graph G, we construct a new graph H by attaching a clique of 4 vertices to every vertex of G. Specifically, for each vertex v in G, the graph H contains three new vertices v_1, v_2, v_3, along with edges $vv_1, vv_2, vv_3, v_1v_2, v_1v_3, v_2v_3$. I claim that

G has a proper 3-coloring
if and only if
H has a slightly improper 3-coloring.

$\Longrightarrow$ Suppose G has a proper 3-coloring, using the colors red, yellow, and blue. Extend this color assignment to the vertices of H by coloring each vertex v_1 red, each vertex v_2 yellow, and each vertex v_3 blue. With this assignment, each vertex of H has at most one neighbor with the same color. Specifically, each vertex of G has the same color as one of the vertices in its gadget, and the other two vertices in v's gadget have no neighbors with the same color.

$\not\Longleftarrow$ Now suppose H has a slightly improper 3-coloring. Then G must have a proper 3-coloring because... um...

♣

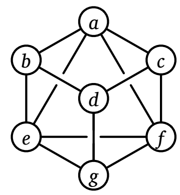
Describe a graph G that does not have a proper 3-coloring, such that the graph H constructed by this reduction does have a slightly improper 3-coloring.

(b) Describe a small graph X with the following property: In every slightly improper 3-coloring of X, every vertex of X has *exactly* one neighbor with the same color.

(c) Describe a correct polynomial-time reduction from 3Color to SlightlyImproper-3Color. *[Hint: Use your graph from part (b) as a gadget.]* This reduction will prove that SlightlyImproper3Color is indeed NP-hard.

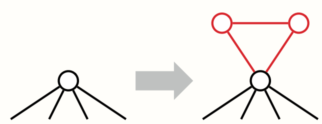
Solved Problem

3. A *double-Hamiltonian tour* in an undirected graph G is a closed walk that visits every vertex in G exactly twice. Prove that it is NP-hard to decide whether a given graph G has a double-Hamiltonian tour.



This graph contains the double-Hamiltonian tour $a \to b \to d \to g \to e \to b \to d \to c \to f \to a \to c \to f \to g \to e \to a$.

Solution: We prove the problem is NP-hard with a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a small gadget to every vertex of G. Specifically, for each vertex v, we add two vertices $v^\sharp$ and $v^\flat$, along with three edges $vv^\flat$, $vv^\sharp$, and $v^\flat v^\sharp$.



A vertex in G and the corresponding vertex gadget in H.

Now I claim that

> G has a Hamiltonian cycle
> if and only if
> H has a double-Hamiltonian tour.

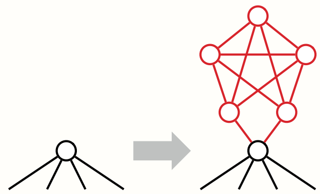
$\Longrightarrow$ Suppose G contains a Hamiltonian cycle $C = v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by replacing each vertex v_i in C with the following walk:

$$\cdots \to v_i \to v_i^\flat \to v_i^\sharp \to v_i^\flat \to v_i^\sharp \to v_i \to \cdots$$

$\Longleftarrow$ Conversely, suppose H has a double-Hamiltonian tour D. Consider any vertex v in the original graph G; the tour D must visit v exactly twice. Those two visits split D into two closed walks, each of which visits v exactly once. Any walk from $v^\flat$ or $v^\sharp$ to any other vertex in H must pass through v. Thus, one of the two closed walks visits only the vertices v, $v^\flat$, and $v^\sharp$. Thus, if we remove the vertices and edges in $H \setminus G$ from D, we obtain a closed walk in G that visits every vertex in G exactly once.

Given any graph G, we can clearly construct the corresponding graph H in polynomial time by brute force.

With more effort, we can construct a graph H that contains a double-Hamiltonian tour ***that traverses each edge of H at most once*** if and only if G contains a Hamiltonian cycle. For each vertex v in G we attach a more complex gadget containing five vertices and eleven edges, as shown on the next page.

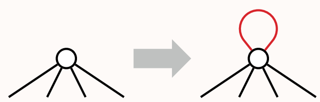


A vertex in G, and the corresponding modified vertex gadget in H.

■

Rubric: 10 points, standard polynomial-time reduction rubric. This is not the only correct solution.

Non-solution (self-loops): We attempt to prove the problem is NP-hard with a reduction from the Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a self-loop every vertex of G. Given any graph G, we can clearly construct the corresponding graph H in polynomial time.



An incorrect vertex gadget.

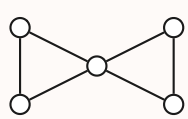
Now I claim that

> G has a Hamiltonian cycle
> if and only if
> H has a double-Hamiltonian tour.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by alternating between edges of the Hamiltonian cycle and self-loops: $v_1 \to v_1 \to v_2 \to v_2 \to v_3 \to \cdots \to v_n \to v_n \to v_1$.

$\not\Longleftarrow$ Um...

Unfortunately, if H has a double-Hamiltonian tour, we *cannot* conclude that G has a Hamiltonian cycle, because we cannot guarantee that a double-Hamiltonian tour in H uses *any* self-loops. The graph G shown below is a counterexample; it has a double-Hamiltonian tour (even before adding self-loops!) but no Hamiltonian cycle.



This graph has a double-Hamiltonian tour.

♣

Some useful NP-hard problems. You are welcome to use any of these in your own NP-hardness proofs, except of course for the specific problem you are trying to prove NP-hard.

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

ChromaticNumber: Given an undirected graph G, what is the minimum number of colors required to color its vertices, so that every edge touches vertices with two different colors?

HamiltonianPath: Given graph G (either directed or undirected), is there a path in G that visits every vertex exactly once?

HamiltonianCycle: Given a graph G (either directed or undirected), is there a cycle in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

LongestPath: Given a graph G (either directed or undirected, possibly with weighted edges), what is the length of the longest simple path in G?

SteinerTree: Given an undirected graph G with some of the vertices marked, what is the minimum number of edges in a subtree of G that contains every marked vertex?

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of $3n$ positive integers, can X be partitioned into n three-element subsets, all with the same sum?

IntegerLinearProgramming: Given a matrix $A \in \mathbb{Z}^{n \times d}$ and two vectors $b \in \mathbb{Z}^n$ and $c \in Z^d$, compute $\max\{c \cdot x \mid Ax \leq b, x \geq 0, x \in \mathbb{Z}^d\}$.

FeasibleILP: Given a matrix $A \in \mathbb{Z}^{n \times d}$ and a vector $b \in \mathbb{Z}^n$, determine whether the set of feasible integer points $\max\{x \in \mathbb{Z}^d \mid Ax \leq b, x \geq 0\}$ is empty.

Draughts: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

SuperMarioBrothers: Given an $n \times n$ Super Mario Brothers level, can Mario reach the castle?