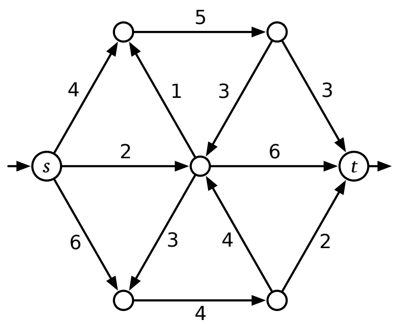
Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. Clearly indicate the following structures in the directed graph on the right. Some of these subproblems may have more than one correct answer.
   (a) A maximum (s,t)-flow f.
   (b) The residual graph of f.
   (c) A minimum (s,t)-cut.



2. Recall that a family $\mathcal{H}$ of hash functions is ***universal*** if $\Pr_{h\in\mathcal{H}}[h(x) = h(y)] \le 1/m$ for all distinct items $x \neq y$, where m is the size of the hash table. For any fixed hash function h, a ***collision*** is an unordered pair of distinct items $x \neq y$ such that $h(x) = h(y)$.

   Suppose we hash a set of n items into a table of size $m = 2n$, using a hash function h chosen uniformly at random from some universal family. Assume $\sqrt{n}$ is an integer.

   (a) ***Prove*** that the expected number of collisions is at most $n/4$.
   (b) ***Prove*** that the probability that there are at least $n/2$ collisions is at most $1/2$.
   (c) ***Prove*** that the probability that any subset of more than $\sqrt{n}$ items all hash to the same address is at most $1/2$. *[Hint: Use part (b).]*
   (d) ***[The actual exam question assumed only pairwise independence of hash values; under this weaker assumption, the claimed result is actually false. Everybody got extra credit for this part.]***

   Now suppose we choose h at random from a ***strongly 4-universal*** family of hash functions, which means for all distinct items w, x, y, z and all addresses i, j, k, l, we have

$$\Pr_{h\in\mathcal{H}}\left[h(w) = i \,\wedge\, h(x) = j \,\wedge\, h(y) = k \,\wedge\, h(z) = \ell\right] = \frac{1}{m^4}.$$

   Prove that the probability that any subset of more than $\sqrt{n}$ items all hash to the same address is at most $O(1/n)$.

   [Hint: Use Markov's and Chebyshev's inequalities. All four statements have short elementary proofs.]

3. Suppose we have already computed a maximum flow f^* in a flow network G with *integer* capacities. Assume all flow values $f^*(e)$ are integers.

   (a) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is *increased* by 1.

   (b) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is *decreased* by 1.

   Your algorithms should be significantly faster than recomputing the maximum flow from scratch.

4. Let T be a treap with n vertices.

   (a) What is the *exact* expected number of leaves in T?

   (b) What is the *exact* expected number of nodes in T that have two children?

   (c) What is the *exact* expected number of nodes in T that have exactly one child?

   You do *not* need to prove that your answers are correct. *[Hint: What is the probability that the node with the kth smallest search key has no children, one child, or two children?]*

5. There is no problem 5.