CS 598: Computational Geometry	Spring 2008

Lecture 02 — January 25

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What Is Geometry?

Klein: Geometry is the study of properties of points, lines, etc. which are invariant under transformations.

Hilbert: A game played with undefined terms and strict rules (A formal system!). Whatever meaning one extracts is at worst a distraction, and at best expresses relations between undefined terms.

2.1 Kinds of Geometry

Euclidean: Invariants of subset of $\mathbb{R}^2$ under:

- Translations
- Rotations
- Uniform Scaling

For instance angles are preserved under the above transformations. But these things are not—distance, slope, distance to origin. Convexity is preserved! And orientation:

$$sgn\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

So the convex hull problem is something to talk about in Euclidean Geometry, however, this is not the most general context.

Projective Geometry:

More generally, we can talk about Projective Transformations, which are "something *really* happening in 3-d."

The *Real Projective Plane*, $\mathbb{R}P^2$ is:
$\mathbb{R}^3/\sim$ where $(a, b, c) \sim (\lambda a, \lambda b, \lambda c)$ iff $\lambda \neq 0$

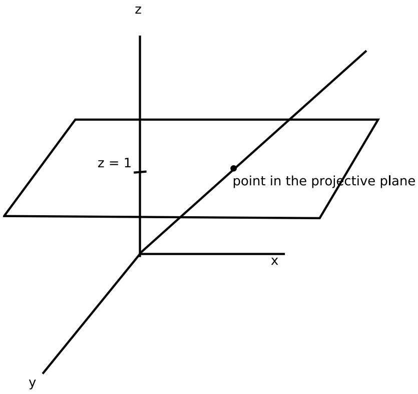

Figure 2.1. A line thru 0 and its intersection in projective plane

We can think of points in $\mathbb{R}P^2$ as "lines through the origin, in $\mathbb{R}^3$". Note that if we took any other point on a line through the origin, except the origin itself, we could rescale it in the projective plane so that the z coordinate is 1.

We have the following implications:

$$\begin{array}{ccc} \text{Euclidean in z} = 1 & \Rightarrow & \text{Non-singular linear transformation of } \mathbb{R}^3 \\ & & \Downarrow \\ & & \text{Projective transformation} \end{array}$$

Two kinds of hulls: $V \subseteq \mathbb{R}^3$
Positive hull, $pos(V) = \{\sum_i \lambda_i v_i | \lambda_i \geq 0 \text{ for all i}\}$
Convex hull, $conv(V) = \{\sum_i \lambda_i v_i | \lambda_i \geq 0 \text{ for all i and } \sum_i \lambda_i = 1\}$

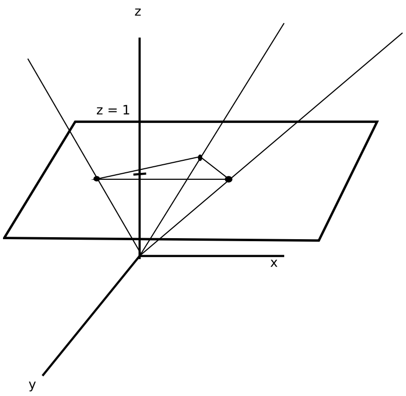

Figure 2.2. A positive hull, and its intersection with the projective plane, a convex hull.

Under these definitions we can think of convex hulls as the intersection of the positive hull with the plane z = 1.

In some sense, convex hulls are invariant under projective transformations. The only problem is, what if you rotate around the y-axis too far (cause it to 'wrap around' at ∞)? Remember the orientation test! The problem occurs because orientation is preserved under positive, but not negative scaling.

Projective transformations preserve "point lies on a line," intersections, but they do not preserve orientations or "point lies above a line."

The solution to this is to think in the *Oriented Projective Plane*: The definition is the same as for the projective plane, but we let, $(x, y, z) \sim (\alpha x, \alpha y, \alpha z)$ iff $\alpha \geq 0$. So points are now *rays* in $\mathbb{R}^3$. We can also think of it as two copies of the plane, one reserved only for positive points, one for negative. However, this interpretation is not the most politically correct.

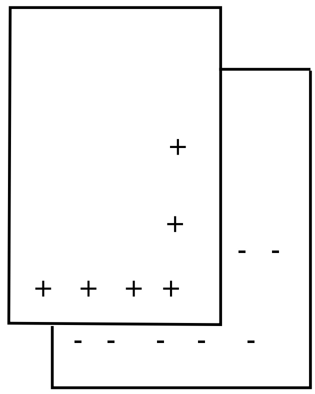

Figure 2.3. Oriented Projective Plane.

We observed that the oriented projective plane is equivalent to the sphere, S^2 (Upper hemisphere are positives, lower are negatives, equator is the line at ∞).

Even in the oriented projective plane, which preserves orientation, we might have a problem if we tilted our positive hull so that some of its rays are parallel to the plane z = 1. But we really were missing a piece of the projective plane and it turns out not to be a problem. We introduce the notion of the *line at* ∞, which is the set of all points of the form $(x, y, 0)$. Note that since z = 0, we can rescale all of these points to $(x, 1, 0)$.

Note also that any line in $\mathbb{R}P^2$ intersects the line at ∞ in two places!

And to complete our variations on hulls, we also have:
affine hull(V) = $\{\sum_i \lambda_i v_i | \sum_i \lambda_i = 1\}$

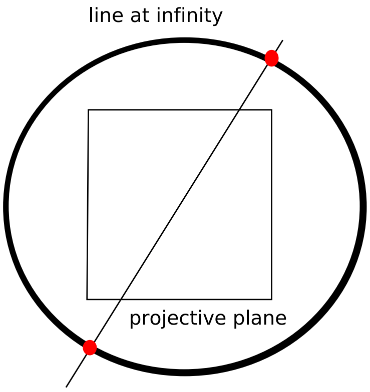

Figure 2.4. A line in the projective plane intersects the line at infinity in two places.

2.2 Meaning, or lack thereof

Consider a variation of geometry where we call:

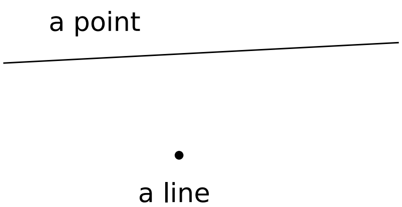

Figure 2.5. That is a fine looking beer mug.

Everything theorem will still be true of the new points and lines! The lesson is that geometry is invariant under name-calling (Galileo knew this about physics!). We remembered that Hilbert said, "It must be possible to replace in all geometric statements the words point, line, plane by table, chair, beer mug."

We can think in a coordinate free space, it doesn't matter so much if it is the Euclidean, Projective, or Oriented Projective plane, we are free to choose coordinates to make the algebra simple.

So we've observed a duality between points/lines and line/points. And that computational geometry is on 1 level geometry and 1 level algebra, but the algebra (or code) may remain the same, there are two geometric interpretations.

Which brings us back again to the point—strip away all the geometric intuition, and all we are left with is:

$$sgn\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}!$$

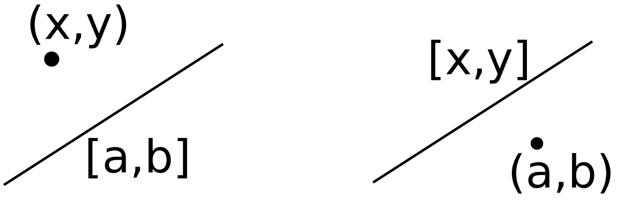

Figure 2.6. Example: $y \geq ax + b$

2.2.1 Duality (Lecture on Jan 30)

Points $\Longleftrightarrow$ Lines $\Longleftrightarrow$ Vectors

Theorem 2.1 (Meta-Theorem). *Any statement about points and lines in the (projective) plane has an equivalent statement about lines and points. In particular, there is an algorithm to convert between the two statements.*

We now develop several isomorphisms:

point $p = (a, b)$	line p^*: $y = ax - b$	p^*: $ax + by = 1$
	slope: a, y-intercept: $-b$	y-intercept: $\frac{1}{b}$, x-intercept: $\frac{1}{a}$
line $l = \overline{pq}$	point $l^* = p^* \cap q^*$	point $l^* = p^* \cap q^*$
$y = ax - b$	(a, b)	
$ax + by = 1$		(a, b)

Duality map is an involution: $f^2 =$ identity

"point on line" $\rightarrow$ "line passing through point"

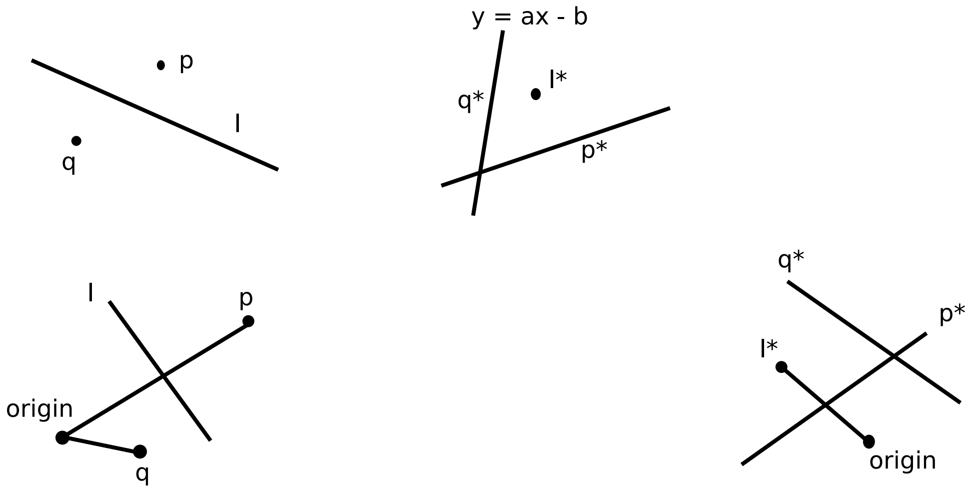

Figure 2.7. Top: Under first duality map, "p above l" $\rightarrow$ "l^* above p^*. Bottom: Under second duality map, "l divides origin and p" $\rightarrow$ "p^* divides origin and l^*

Geometric relations again reduce to algebra: $y - (ax + b) = ?$

(a, b)	$y = ax - b$	$ax + by = 1$
point at ∞	vertical line	line through origin

General position assumptions—if you apply the dualities in an algorithm, you are assuming:

1'st duality: non-vertical lines.

2'nd duality: affine lines only.

You can assume WOLOG no lines are vertical, since we have only finitely many, just rotate a little. Three colinear points dualizes to three lines which are concurrent. So general position in the dual may state that no three lines are concurrent.

We know mention some algebra related to convex hulls in the dual. *All this is under the* first *duality map!* The *Upper Envelope* of a set of lines L, is the set of points P, such that $\forall p \in P\ \forall l \in L, p$ is above l. The Lower Envelope is definied similarly, except instead of the condition "p above l" we use "p below l." Orientation is preserved, so if points p, q, r are oriented counter-clockwise, then in the inner cell of lines p^*, q^*, r^*, a circle going from $p^* \to q^* \to r^*$ will be oriented counter-clockwise.

The relation between a convex hull and its dual is summarized as:

point inside convex hull	line below upper envelope, above lower envelope
line above convex hull	point in lower envelope
line tangent to convex hull	point on envelope (lower or upper)
edge of convex hull (line joining two points)	vertex of envelope (intersection of two lines)
line forming an edge of lower envelope	point on top of CH

Note that there is a one-to-one correspondance between edges of convex hull and vertices of the envelopes.

Under the second *duality map.* Note also that we are assuming that the origin lies inside the convex hull we are considering. We will denote the *Inner Envelope* of a set of lines L, to be the set of points P, such that $\forall p \in P\ \forall l \in L, l$ does not divide the origin and p. Under this map, lines outside the hull will determine points in the inner cell (the cell containing the origin). We will again have a one-to-one correspondance between edges of convex hull and vertices of the envelopes.

From this duality map, we get another hull algorithm:

Given a set of points, find a set of lines with the property that each line passes through 2 points, and have all the other points on one side of the line. We can also find the points given the lines.

So we also have a notion of algorithmic duality. The algorithm to find the inner envelope of a set of lines, is the same as to find the convex hull of a set of points. The only thing we need to worry about is to aviod degenerate cases, but as we have seen we can make general position assumptions, like "no lines are through the origin."

We ended with the remark that the picture of a convex hull and its dual are combinatorially equivalent, and the reason is that they are projective transofrmations of each other.