Lecture 4 — January 30

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4.1 Voronoi Diagrams

4.1.1 Introduction and Definition

Voronoi diagrams have been invented and reinvented numerous times and in numerous contexts. They can be found as early as Descartes's analysis of cosmic fragmentation, but their first careful definition is usually attributed to either Voronoi or Dirichlet. The name Theissen has sometimes been attached to them by meteorologists, and physicists may know them by any of the names Wigner, Seitz, or Broullin. Biologists have also rediscovered the Voronoi diagram; Brown discussed them in 1965, and Mead described them in 1966. Suffice it to say that the Voronoi diagram is an important and useful object.

The formal definition of a Voronoi diagram is as follows. Given a set P of points, called *sites* to distinguish them from arbitrary points, the Voronoi diagram is a subdivision of space into the cells

$$\mathrm{Vor}(p, P) = \{x : |px| \leq |qx| \quad \forall q \in P\},$$

with one cell for each site $p \in P$. Fig 4.1 gives a graphical representation of a Voronoi diagram and provides labels for the important parts of the Voronoi diagram.

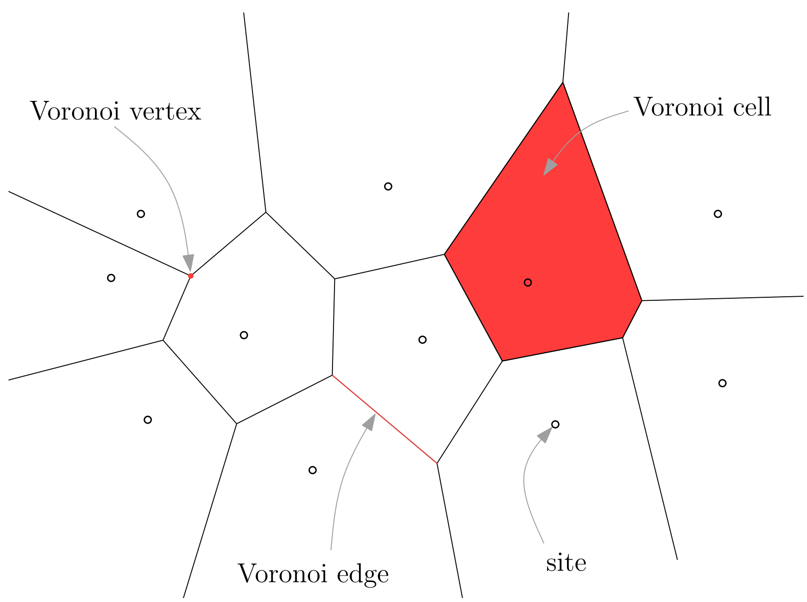

Figure 4.1. A Voronoi diagram, with component parts labeled.

The boundaries of the Voronoi diagram in Fig. 4.1 appear to be and, in fact, are line segments. Every cell boundary in a standard Voronoi diagram is a line segment, possibly of infinite length. This can be verified algebraically by solving the equation that sets equal the (squared) distance to each of two sites. One can also prove this using Euclid's compass and straight-edge construction of the perpendicular bisector. (See Fig. 4.2.) Euclid proved that one could construct the perpendicular bisector of segment pq by using the compass to construct two circles of equal but arbitrary radius $r > |pq|/2, r \neq |pq|$ and using the straight-edge to construct the line that passed through the two intersections of the pair of circles.

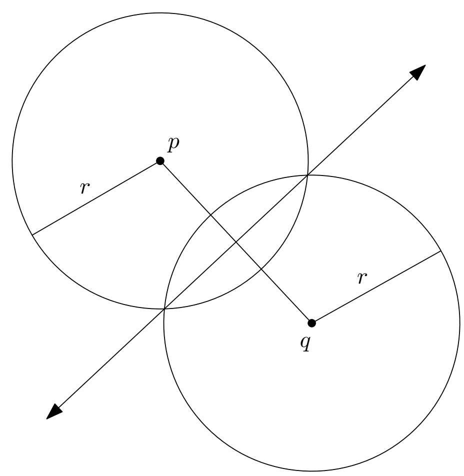

Figure 4.2. Euclid's construction of the perpendicular bisector.

Euclid also showed that the three perpendicular bisectors of the edges of a triangle meet each other at the circumcenter of the triangle — the unique point that is at equal distance from every vertex of the triangle. (See Fig. 4.3.) This means that we can compute the vertices of the Voronoi diagram by computing the circumcenters of certain triangles. That's good news, because computing the circumcenter of a triangle reduces to solving a linear system, something computers can do efficiently.

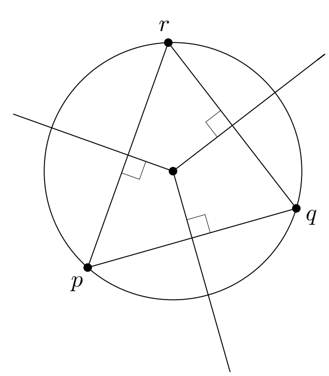

Figure 4.3. The perpendicular bisectors meet at the circumcenter.

Before moving on, we note that all of the circles passing through points p and q have centers that are equidistant from p and q. This means that the circle centers all lie on the perpendicular bisector of p and q. In the Voronoi diagram we can imagine sliding a circle center along a Voronoi edge. The circle will shrink or grow as we move towards or away from segment pq. If the Voronoi edge is finite we eventually reach a Voronoi vertex as we move the circle center along the Voronoi edge. This marks the event of our circle touching a new third vertex. Observe that for infinite Voronoi edges we can move the circle center along the Voronoi edge all the way to a point at infinity without our circle colliding with a third vertex. When the circle center reaches infinity, our circle becomes a line, and the statement that the circle is empty becomes the statement that all of P lies on the same side of that line. In other words, the points that correspond to an infinite Voronoi edge are points that lie on the convex hull of P. This fact will be useful when we design algorithms to compute the Voronoi diagram.

4.1.2 Incircle Test, The Crucial Geometric Primitive

We know how to compute the Voronoi diagram of two vertices (perpendicular bisector) and of three vertices (three perpendicular bisectors meeting at the circumcenter). Now we consider how to compute the Voronoi diagram of four vertices. If the four vertices are in general position, meaning they are not all cocircular or colinear, then finding the Voronoi diagram of four vertices boils down to choosing between the two possible configurations shown in Fig. 4.4

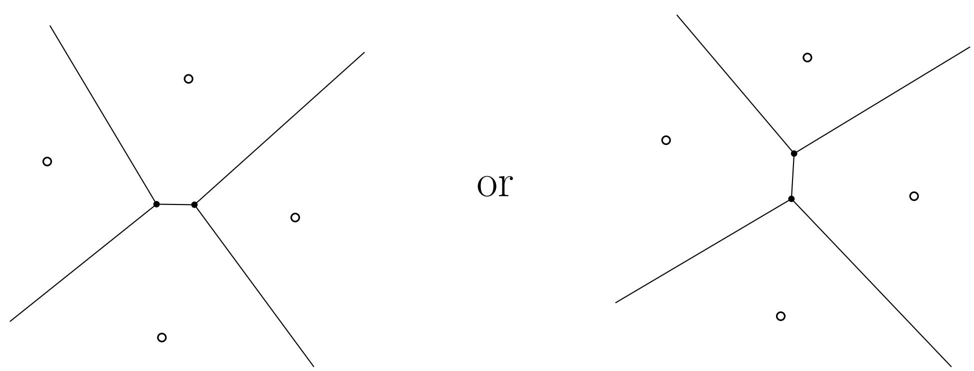

Figure 4.4. The basic decision in computing the Voronoi diagram.

In fact, this decision is determined by what is called the *incircle test*, the test of whether a fourth point is inside or outside of the circle that the first three points determine. Consider four points p, q, r, and s. Figure 4.5 shows the Voronoi diagram of points p, q, and r overlaying the Voronoi diagram of points p, r, and s in two different situations. It should be clear from the picture that if s is inside the circle on p, q, and r, then the four-point Voronoi diagram should choose the circumcenters of triangles pqs and qrs, but if s is outside the circle on p, q, and s, then the Voronoi diagram should include the circumcenters of triangles pqr and prs.

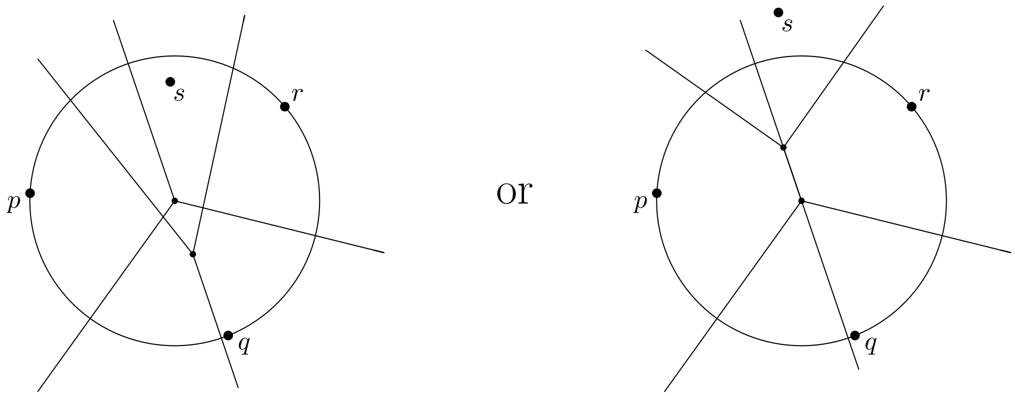

Figure 4.5. The basic decision is made based on the incircle test.

As is true of almost all geometric primitives, the incircle test can be computed algebraically. The test depends on whether

$$\det(A) = \det \begin{pmatrix} 1 & p_x & p_y & p_x^2 + p_y^2 \\ 1 & q_x & q_y & q_x^2 + q_y^2 \\ 1 & r_x & r_y & r_x^2 + r_y^2 \\ 1 & s_x & s_y & s_x^2 + s_y^2 \end{pmatrix} \lesseqgtr 0.$$

Notice that the test of whether s is inside the circle on p, q, and r is the same as the test of whether q is inside the circle on p, r, and s. It's also the same as the test of whether p is *outside* the circle on q, r, and s and the test of whether r is *outside* the circle on p, q, and s.

The comparison of the determinant to 0 does not directly determine whether s is inside or outside of the circle on p, q, and r. It tells whether s is on the left or right as one traverses the circle from p to q to r. Thus if p, q, and r are oriented clockwise and s is on the left ($\det(A) < 0$), that means s is outside of the circle on p, q, and r. On the other hand, if p, q, and r are oriented counterclockwise and s is on the left ($\det(A) < 0$), then s is inside the circle. The upper left 3×3 submatrix of matrix A is the matrix used in the orientation test for p, q, and r, so very little additional computation is necessary, but when implementing the incircle test, one must be careful to do this correctly.