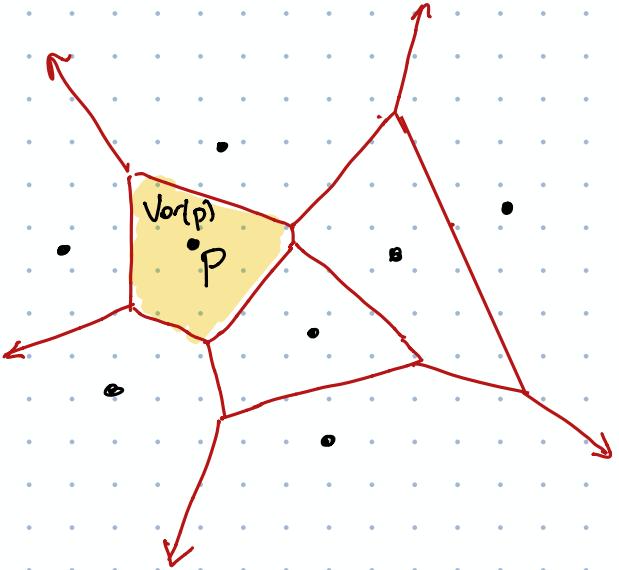


# Voronoi Diagrams

Das Reckst - "Combination Pizza Hut and Taco Bell"  
 Strocolab - "Diagonals"  
 Alt-J( $\Delta$ ) - "Tessellate"



$P = \text{Finite point set} - \text{"sites"}$

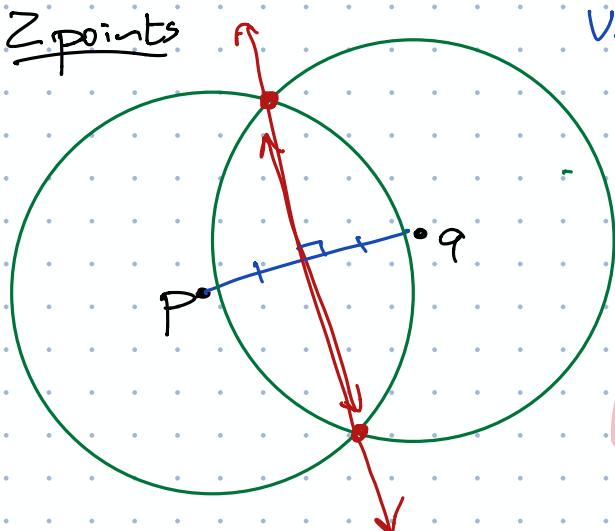
Voronoi region  $\text{Vor}(p)$

$$= \{q \in \mathbb{R}^2 \mid p \text{ is the closest site to } q\}$$

## Applications:

- Nearest neighbor queries
- Interpolation

Fortune's Algorithm  $\sim O(n \log n)$  time



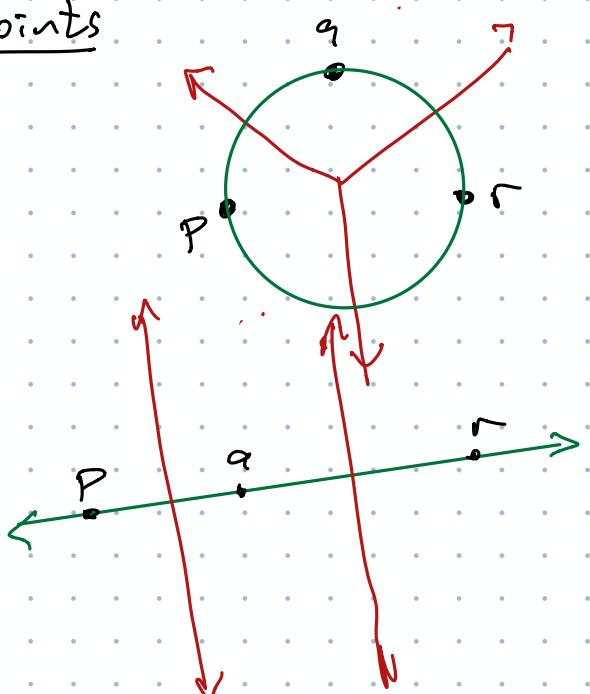
Voronoi edge = perp. bisector of  $pq$

$$\|(b, b) - (x, y)\|^2 = \|(c, d) - (x, y)\|^2$$

$$\begin{aligned} & a^2 - 2ax + x^2 + b^2 - 2by + y^2 \\ & = c^2 - 2cx + x^2 + d^2 - 2dy + y^2 \end{aligned}$$

$$(a-c)x + (b-d)y = \frac{1}{2}(a^2 + b^2 - c^2 - d^2)$$

3 points



Voronoi vertex =

Circumcenter of 3 sites  
 $\downarrow$   
 center of circumcircle

General position:  
 No 3 pts collinear  
 No 4 pts cocircular

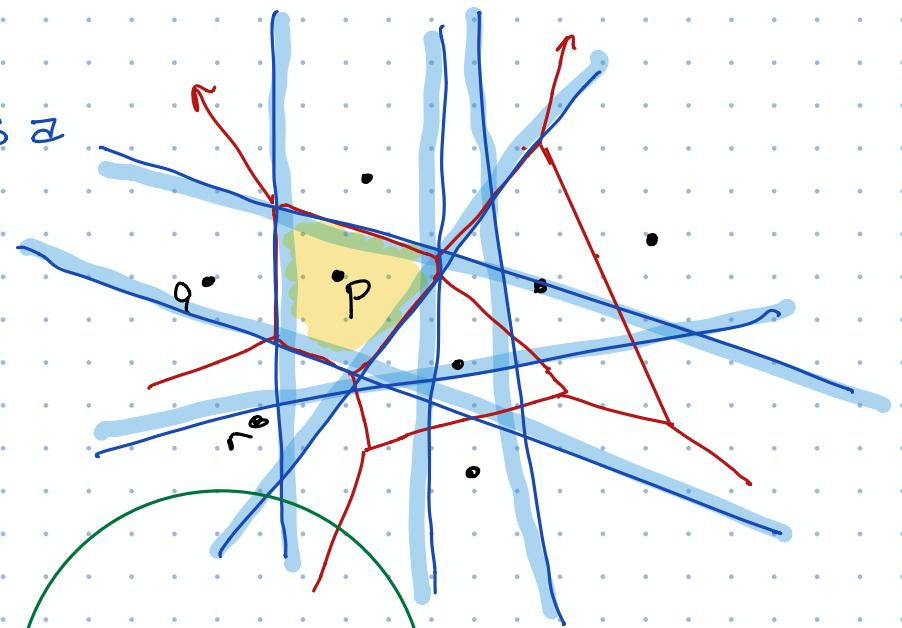
More generally:

Every Voronoi region is a convex polygon

↳ intersection  
of  $n-1$  bisection  
halfplanes

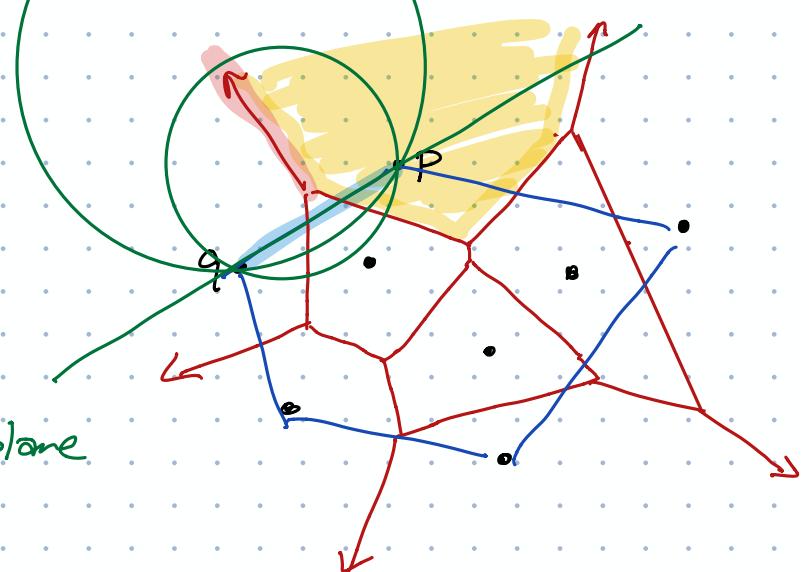
→  $\text{Vor}(p)$  connected

→ Compute  $\text{Vor}(p)$   
in  $O(n \log n)$  time



- $\text{Vor}(p)$  is unbounded iff  $p$  is a vertex of  $\text{conv}(P)$
- Infinite Voronoi edges  $\iff$  convex hull edges

Growing empty circle  
becomes empty halfplane  
in the limit



Planar straight-line graph, every vertex has degree 3

Euler's formula

$$\Rightarrow \begin{array}{l} n \text{ faces} \\ 2n - 2 - h \text{ vertices} \\ 3n - 3 - h \text{ edges} \\ \text{where } h = \# \text{ hull vertices} \end{array}$$

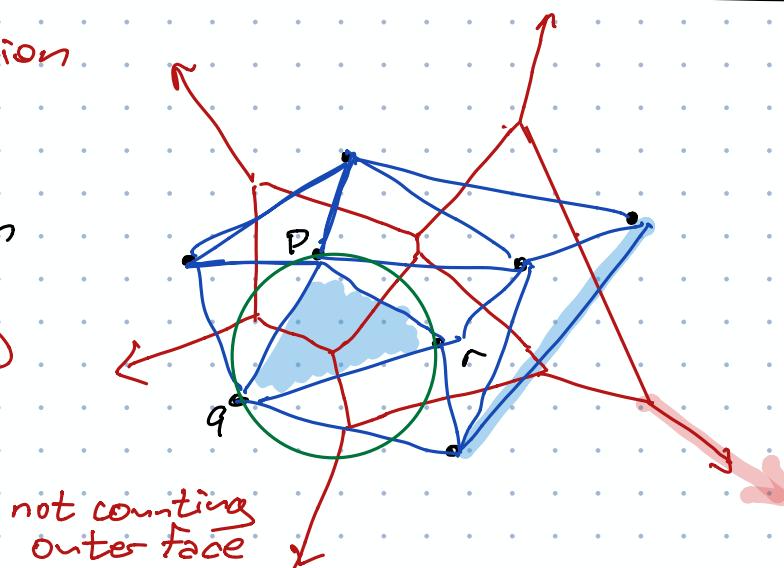
not counting sentinel vertex at  $\infty$

## Delannay Triangulation

straight-line  
= planar dual of  
Voronoi diagram

Triangulation of  $P$   
(assuming gen-pos.)

PSLG     $n$  vertices  
           $3n - 3 - h$  edges  
           $2n - 2 - h$  faces



Standard DS for Voronoi = Standard DS for Delaunay

$pq$  is a Delaunay edge iff some circle has  $p, q$  inside all other sites outside

$pqr$  is a Delaunay triangle iff circumcircle ( $p, q, r$ ) is empty

Circles!



$$(x-a)^2 + (y-b)^2 = r^2$$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

$$\text{Of } \text{coeff}(x^2) = \text{coeff}(y^2) \quad \text{coeff}(xy) = 0$$

$$x^2 + y^2 - 2\alpha x - 2\beta y + \gamma = 0$$

circle with center  $(\alpha, \beta)$  radius  $\sqrt{\gamma - \alpha^2 - \beta^2}$

Circumcircle of  $(a, b), (c, d), (e, f)$ :

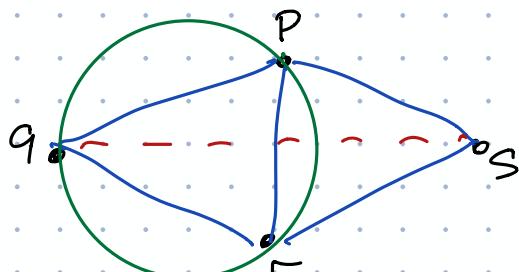
$$\det \begin{bmatrix} 1 & a & b & a^2 + b^2 \\ 1 & c & d & c^2 + d^2 \\ 1 & e & f & e^2 + f^2 \\ 1 & x & y & x^2 + y^2 \end{bmatrix} = 0$$

$$= (x^2 + y^2) \begin{vmatrix} 1 & a & b \\ 1 & c & d \\ 1 & e & f \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 + b^2 \\ 1 & c & c^2 + d^2 \\ 1 & e & e^2 + f^2 \end{vmatrix} + x \left| \begin{array}{c} a^2 + b^2 \\ c^2 + d^2 \\ e^2 + f^2 \end{array} \right.$$

degenerates to a line if pts are collinear

If  $\det > 0$  and 3pts are CCW  
( $x, y$ ) inside circle

4 points



$s$  outside  $\text{circ}(pqr)$

$$\begin{vmatrix} 1 & p_x & p_y & p_x^2 + p_y^2 \\ 1 & q_x & q_y & q_x^2 + q_y^2 \\ 1 & r_x & r_y & r_x^2 + r_y^2 \\ 1 & s_x & s_y & s_x^2 + s_y^2 \end{vmatrix} > 0$$

InCircle test ( $p, q, r, s$ )

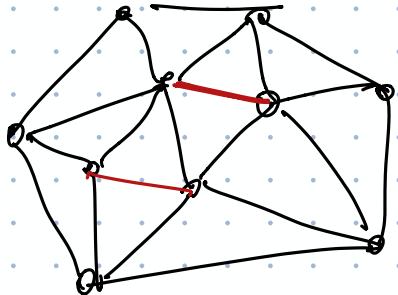
A triangulation is locally Delaunay if every pair of adjacent  $\Delta$ s passes the incircle test  
"Every interior edge is locally Delaunay"

Theorem: Delaunay = locally Delaunay

Proof on Thursday

Lawson's flip algorithm (1977)

- ① Triangulate  $P$
- ② Repeatedly flip bad edges until all edges are locally Delaunay



Theorem  $\Rightarrow$

IF algo halts,  
final triangulation  
is Delaunay

- It does halt
- After  $O(n^2)$  Flips
- Basis of  $O(n \log n)$ -time randomized incremental algorithm