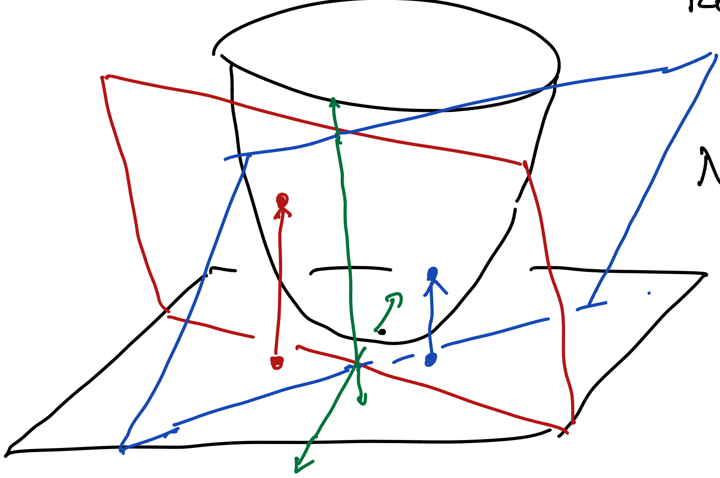
Weighted Voronoi/Delaunay and 3D convex hulls

So far we have three equivalent structures:
- Voronoi diagrams
- Delaunay triangulations
- Lower convex hulls of points on paraboloid $z=(x^2+y^2)/2$

There is a fourth!



$\pi(x,y) = (x^2+y^2)/2$

$\Downarrow$

$\nabla\pi(x,y) = (x,y)$!

Recall lifting map

$\underset{p}{(a,b)} \mapsto \underset{\hat{p}}{(a,\, b,\, (a^2+b^2)/2)}$

Now add duality!

$(a,b,c) \mapsto z = ax+by-c$

$\hat{p} \longmapsto \hat{p}^*$

$\hat{p}^*$ is the plane tangent to the paraboloid at point $\hat{p}$

$(a^2+b^2)/2 = a\cdot a + b\cdot b - (a^2+b^2)/2$

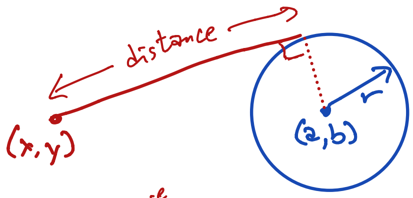
$\boxed{\text{The Voronoi diagram of } P \text{ is the projection of the lower envelope of the planes } \hat{p}^*}$

$\mathbb{R}^3$	Lower convex hull (empty halfplanes)	$\xleftrightarrow[\text{duality}]{\text{projective}}$	Upper envelope (highest plane
	lift to points on paraboloid $\uparrow\downarrow$ projection		lift to planes tangent to paraboloid $\uparrow\downarrow$ projection
$\mathbb{R}^2$	Delaunay triangulation (empty circles)	$\xleftrightarrow[\text{duality}]{\text{planar map}}$	Voronoi diagram (nearest neighbor)

- What about points off the paraboloid?
  - $\Rightarrow$ weighted Voronoi diagrams (Laguerre, power)
  - $\Rightarrow$ weighted Delaunay triangulations (regular, coherent)
- What about the upper hull / lower envelope?
  - $\Rightarrow$ anti-Voronoi, anti-Delaunay (furthest-point)

Power diagrams

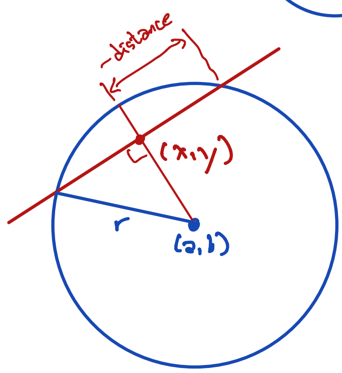
Think of a weighted point as a circle



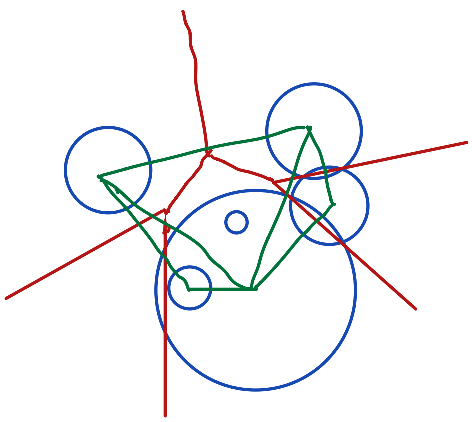
The (squared) power distance from a point (x,y) to a circle with center (a,b) and radius r

is $(x-a)^2+(y-b)^2-r^2$

>0 outside
$=0$ on
<0 inside



The power diagram of a set of circles partitions ~~the~~ plane by which circle has minimum power distance

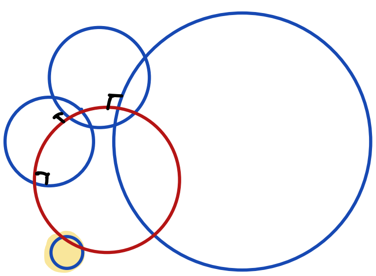


Bisector = line through intersection points

Some power regions can be empty!

Dual = weighted Delaunay triangulation

Not all sites are vertices



Is C_4 "outside" the orthocircle of C_1, C_2, C_3?

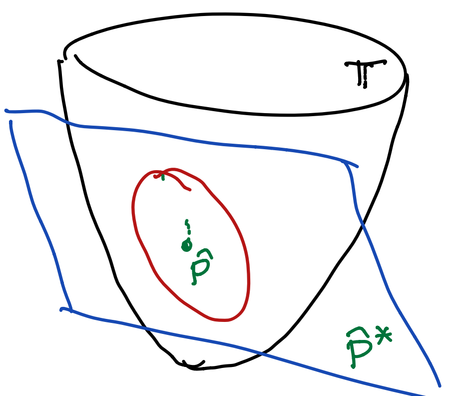
Equivalent of incircle test:

$$\operatorname{sgn}\det\begin{bmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2-r_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2-r_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2-r_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2-r_4^2 \end{bmatrix}$$

Lift to points in $\mathbb{R}^3$:

$$((a,b),r) \rightsquigarrow \left(a,\, b,\, \frac{a^2+b^2-r^2}{2}\right)$$

Weighted Delaunay triangulation = projection of lower hull



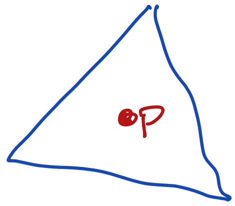
$\pi \cap \hat{p}^*$ = visibility horizon of $\hat{p}$

Lift to planes in $\mathbb{R}^3$

$$((a,b),r) \rightsquigarrow z = ax+by-\frac{a^2+b^2-r^2}{2}$$

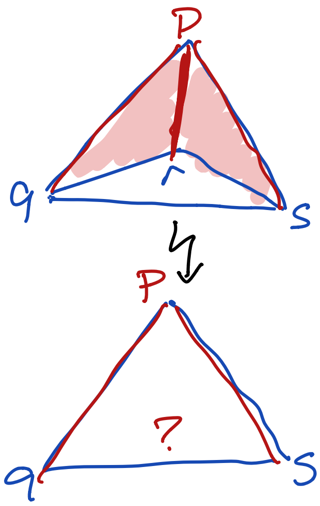
Power diagram = projection of upper envelope

Construct using same rand-incr. flip algorithm with two minor changes:



If p is inside qrs,

check whether $\hat{p}$ is above $\hat{q}\hat{r}\hat{s}$ before trisecting



If two adjacent triangles fail weighted incircle test and their union is non-convex, delete the reflex vertex

In both of these cases, we are discovering a point above the lower convex hull.

Still worst case $O(n^2)$ time
expected $O(n \log n)$ time

Anti-Voronoi / Anti-Delaunay

↳ Partitions the plane by furthest site

↳ Every triangle has a full circumcircle

Lift to points in $\mathbb{R}^3$: $(a,b) \mapsto (a, b, \frac{a^2+b^2}{2})$

Anti-Delaunay = projection of upper hull

Lift to planes in $\mathbb{R}^3$: $(a,b) \mapsto z = ax + by - \frac{a^2+b^2}{2}$

Anti-Voronoi = projection of upper envelope

Same algorithm again, but reverse sign of all incircle tests

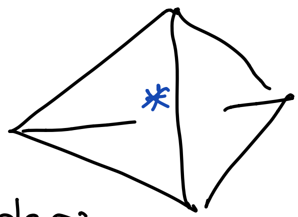
(Equivalently, negate z-coords and treat as weighted Voronoi / Delaunay)

3D convex hulls

Really all these are special cases of the convex hull problem in $\mathbb{R}^3$.

We can recast the randomized incremental algorithm as follows:

Start with any 4 points:

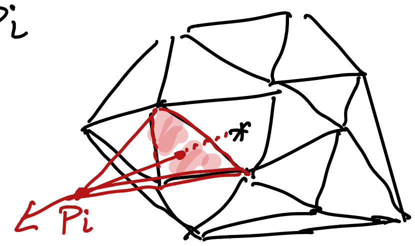


Let $\circ$ be any interior point

Insert points in random order:

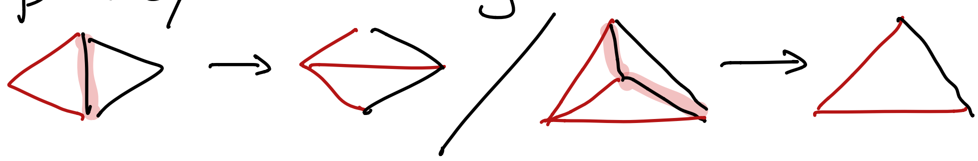
(1) Find a facet visible to p_i

Facet intersected by $\overrightarrow{op_i}$ using history dag



If p_i "inside" that facet, we're done

(2) Connect p_i to visible facet
Flip away nonconvex edges + vertices



Everything relies on 3d orientation tests

$$\operatorname{sgn}\det\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \\ 1 & j & k & l \end{pmatrix}$$

Same analysis $\longrightarrow$ $O(n\log n)$ exp. time

In higher dimensions:

$$O(n\log n + \underbrace{n^{\lfloor d/2 \rfloor}}_{\text{worst case output size}})\text{ exp. time}$$

(More types of "flip")