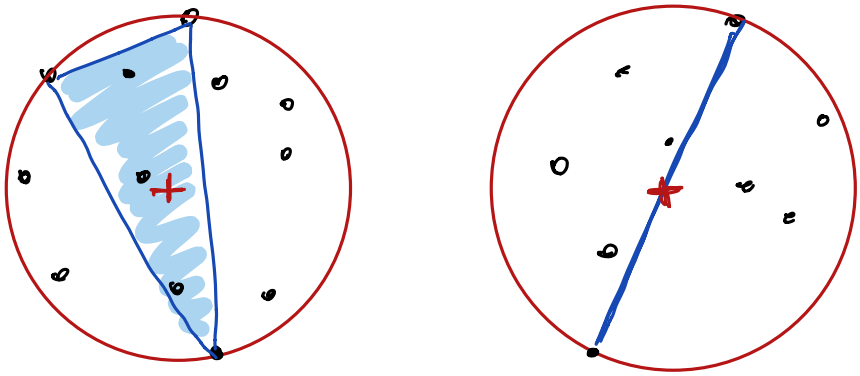
Smallest enclosing ball "Smallest bomb problem"

$$\boxed{\begin{array}{l} \min\ r^2 \\ \text{s.t. } (x_i-a)^2+(y_i-b)^2 \le r^2 \end{array}}$$

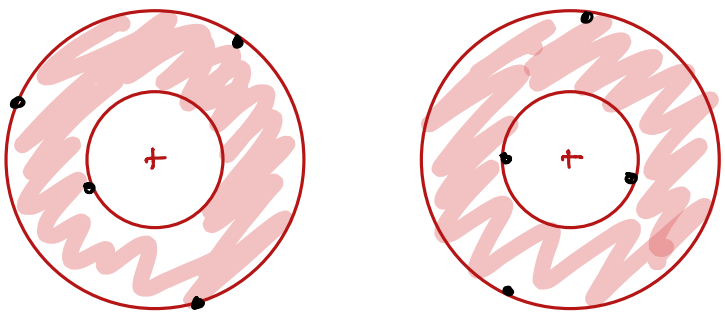
(a,b) center
r radius

Non-linear constraints can't be linearized

The optimum satisfies either 2 or 3 constraints with equality



Compare smallest annulus:



Because it's the solution to an LP with 4 variables,

Optimum annulus must have 4 points on its boundary.

But Seidel's algorithm doesn't really depend on linearity.

- Solution is unique (assuming gen pos)
- Solution is determined by tight constraints
- Exchange property: If $OPT(H-h, B) \notin h$ then $OPT(H,B) \in \partial h$.

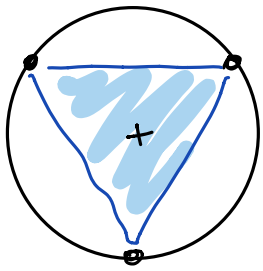
Let's prove these properties for smallest enclosing circle

Uniqueness follows from convexity:

If P lies inside $B(c,r)$, then $c \in \underbrace{\bigcap_{p \in P} B(p,r)}_{\text{Call this } P_r}$

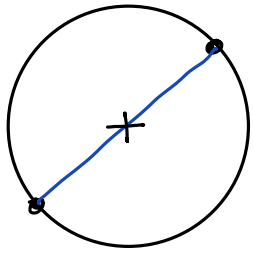
P_r is convex for all r
shrinks to a point as r increases
(can't be a segment = convex curve)

Solution determined by tight constraints:



Circumcircle of pqr = smallest circle pqr

iff center inside pqr
iff pqr is <u>acute</u>



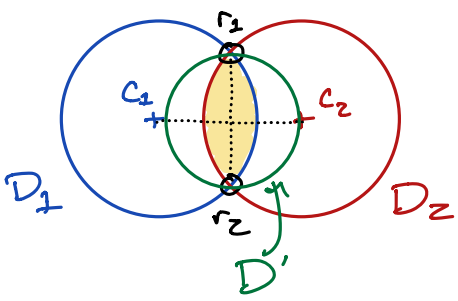
smallest circle containing pq
centered at midpoint

(Again, convexity!)

<u>Lemma</u>: Let P = pts in plane
R = pts in plane $\quad P \cap R = \emptyset$

$\boxed{\text{If there is a disk } D \text{ with } P \subseteq D \text{ and } R \subseteq \partial D \text{ the smallest such disk is unique.}}$ $\leftarrow \min D(P,R)$

<u>Proof</u>: Let D_1 and D_2 be two such minimal disks $\Rightarrow |R| \leq 2$



$P \subseteq D_1$ and $P \subseteq D_2 \Rightarrow P \subseteq D_1 \cap D_2$

Let c_1, c_2 be centers of D_1, D_2
$r_1, r_2 = \partial D_1 \cap \partial D_2$

$\boxed{R \subseteq \{r_1, r_2\}}$

Let c' = midpoint of $c_1c_2 = c_1c_2 \cap r_1r_2$

Let D' = disk centered at c' thru r_1r_2

① $\text{radius}(D') < \text{radius}(D_1) = \text{radius}(D_2)$

② $P \subseteq D_1 \cap D_2 \subseteq D'$

So D' is smaller disk with $P \subseteq D'$, $R \subseteq D'$
so D_1, D_2 not smallest $\quad \square$

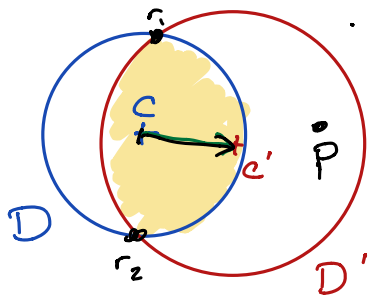
<u>Pivoting Lemma</u>: Let P, R be disjoint point sets as above.

For any $p \in P$: ① If $p \in \min D(P-p, R)$ then $\min D(P,R) = \min D(P-p, R)$
② If $p \notin \min D(P-p, R)$ then $\min D(P,R) = \min D(P-p, R+p)$

Proof: Let $D = \text{minD}(P-p, R)$

① Spose $p \in D$. Then $P \subseteq D$ and $R \subseteq \partial D$
Any smaller disk D' with $P \subseteq D'$ and $R \in \partial D'$ would also have $P-p \in D'$ contradicting def. of D. ✓

② Let $D' = \text{minD}(P, R)$ $\quad P-p \subseteq D \cap D'$



Continuously deform D into D' keeping R on the boundary moving center along the ray $\overrightarrow{cc'}$

D_t always contains $D \cap D'$

$\Rightarrow$ radius must <u>increase</u> over time

$\Rightarrow$ D' is <u>first</u> disk in this family to contain p $\square$

Welzl's minidisk algorithm:

$\text{MinD}(P, R)$: $\Leftarrow |R| \le \cancel{3}\, d+1$

in practice: if $|P \cup R| \le \cancel{3}\, d+1$

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if |R| > ~~3~~ d+1
    return INFEASIBLE
else if P = ∅
    compute MinD(∅, R) by brute force
else
    p ← random point in P
    D ← MinD(P − p, R)
    if p ∈ D
        return D
    else
        return MinD(P − p, R + p)
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Works in arbitrary dimensions!

It's instructive to trace thru this algorithm when $d=2$, $|P|=3$, $R=\emptyset$

Runs in $O((d+2)!\, n)$ expected time (via Seidel analysis)

Similar algorithms work for other <u>LP-type</u> problems:

Monotonicity: $A \subseteq B \Rightarrow OPT(A) \leq OPT(B)$

Locality: $A \subseteq B \Rightarrow \big(OPT(A) = OPT(B) = OPT(A+x) \Rightarrow OPT(B) = OPT(B+x) \big)$

- Minimum enclosing ellipsoid $(D = d(d+3)/2)$
- Quasi-convex optimization: $(D \leq 2d+1)$

  Find min. value of a function with convex sub-level sets

On the other hand,

slow when d is large $\longrightarrow$ use variant of the simplex method instead!