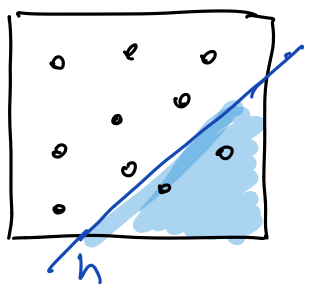
# Applications of line arrangements

(Halfplane) Discrepancy:




Let $P$ be a set of $n$ points in the unit square $[0,1]^2$

For any halfplane $h$, define

$\mu(h) =$ area of $h \cap [0,1]^2$

$\mu_P(h) = |P \cap h| / |P|$

h and $\bar{h}$ have same discrepancy!

discrepancy $\Delta_P(h) = |\mu(h) - \mu_P(h)|$

discrepancy of $P = \sup_h \Delta_P(h)$

Motivated by ray-tracing/antialiasing, Monte Carlo integration

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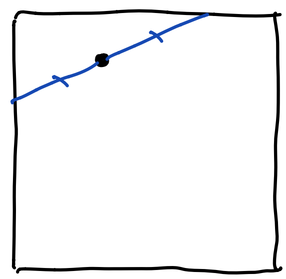
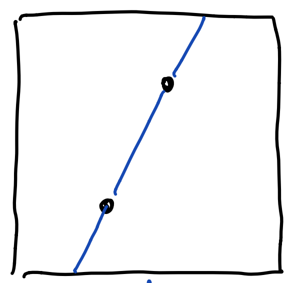
Continuity arguments imply line $\ell$ with max discrepancy has one of two forms:



Through one point $p \in P$ at midpoint of $\ell \cap [0,1]^2$

Only $O(n)$ of these ($\leq 8$ per point in $P$)



Through two points in $P$

But $O(n^2)$ of these.

We can compute $\mu(h)$ in $O(1)$ time

But what about $\mu_P(h)$?

Given set $P$ of $n$ points in $\mathbb{R}^2$,
Find # points in $P$ above each line thru two points in $P$

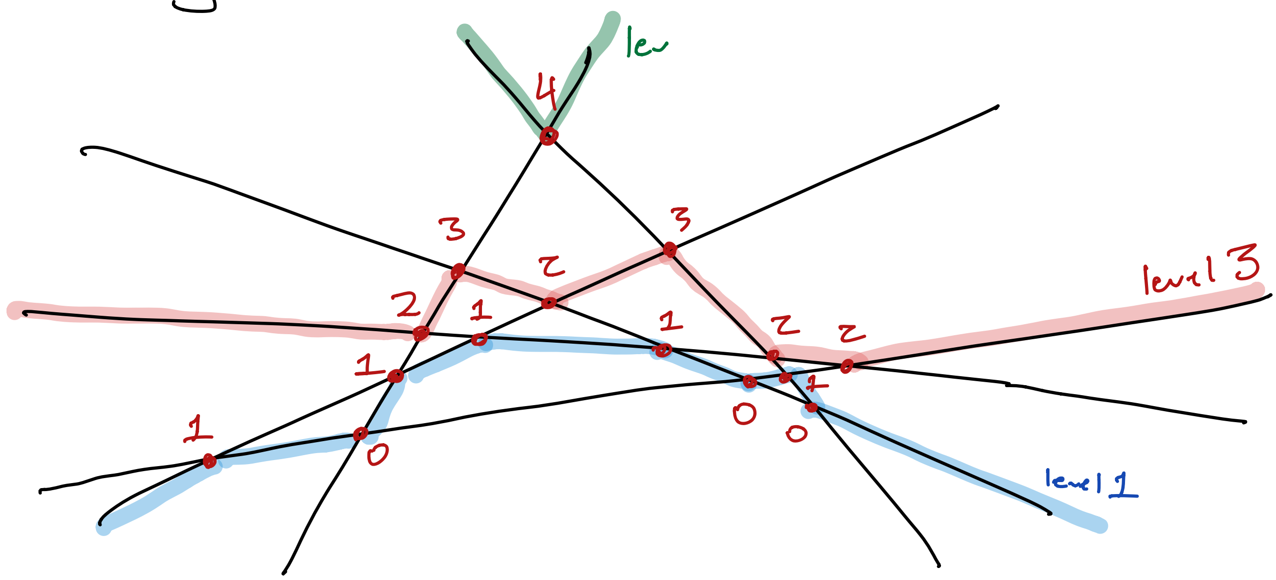
Duality FTW!

$\Downarrow$

Given set $L$ of $n$ lines in $\mathbb{R}^2$,
Find # lines in $L$ below each point on two lines in $L$

Compute the level of every vertex of arrgh($L$)

Build arrgh($L$).




Compute level of any vertex by brute force

Compute remaining levels by WFS

Alternatively: Compute left "end" of each level by sorting slopes

Walk along each level in $O(1)$ time per vertex

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## Ham-Sandwich Cuts

Given two sets of points $R$ and $B$ in the plane,
Find a line that bisects both sets

(3D)

(Ham-sandwich theorem: There is a plane that cuts both slices of bread and the ham exactly in half

even if one slice of bread is on your head and the other one is the moon.)

## Existence proof (2D):

Assume both sets have odd # points

Let $\ell_B$ and $\ell_R$ be unique vertical lines that bisect $B$ and $R$

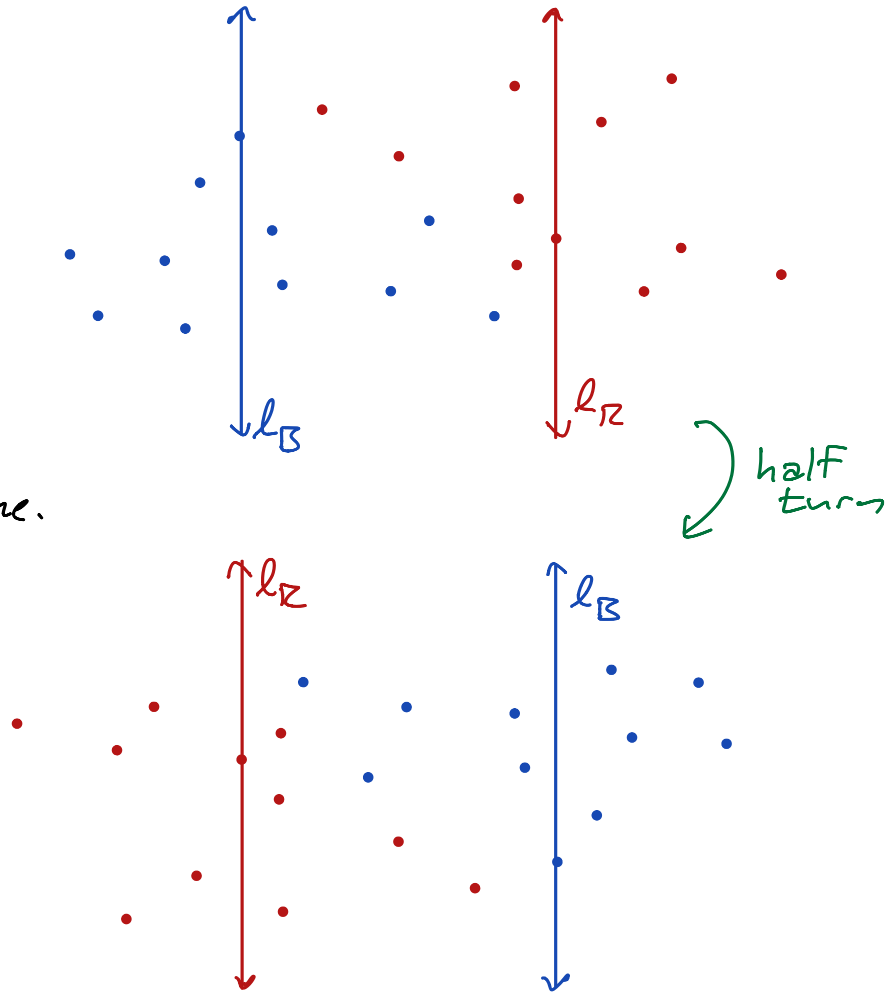
WLOG $\ell_B$ is left of $\ell_R$.

Continuously rotate the plane.

The vertical bisecting lines change continuously!

After half turn, $\ell_B$ is right of $\ell_R$.

So at some angle, they must coincide! □




## Algorithm:

Keep points fixed, vary slopes of bisectors from $-\infty$ to $\infty$

In the dual: Two families of lines $B^*$ and $R^*$

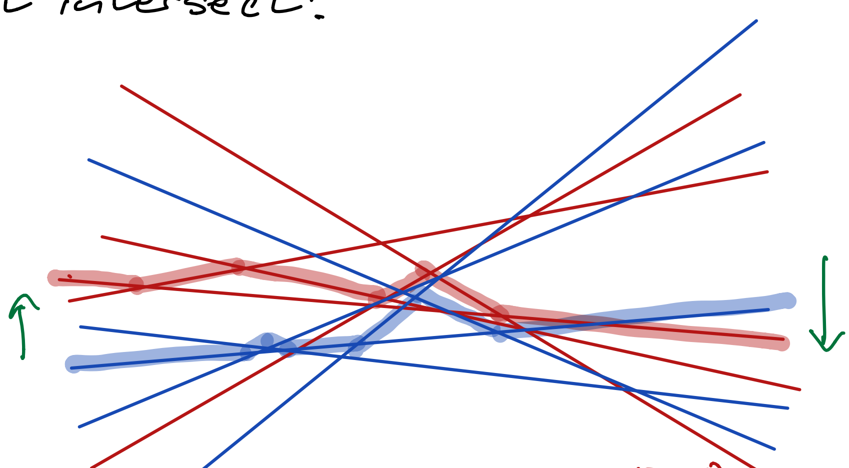
move points $\ell_B^*$ and $\ell_R^*$ from left to right along median levels of arrgh($B^*$) and arrgh($R^*$)

These levels must intersect!

Construct arrangements of $B^*$ and $R^*$

Walk along median levels to find intersection

$O(n^2)$ time



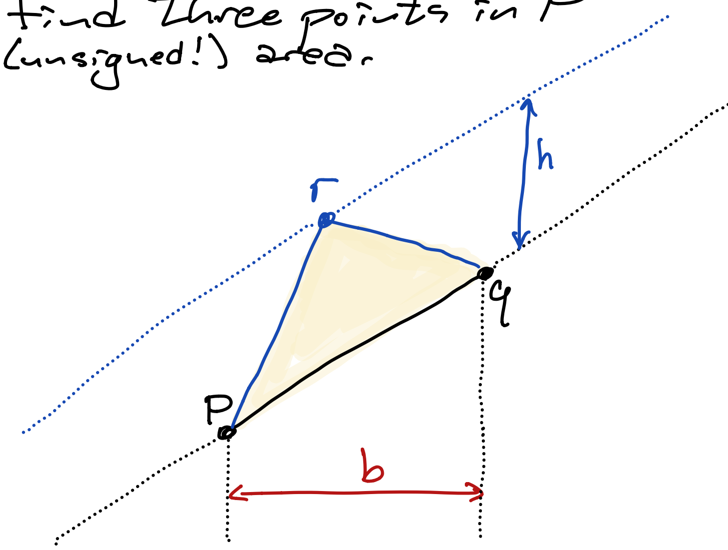
$O(n^{4/3})$ $n \cdot 2^{\Omega(\sqrt{\log n})}$

→ We don't actually know how long this takes!!

## Minimum-area triangles:

Given a set $P$ of $n$ points, find three points in $P$ spanning minimum (unsigned!) area.

Naive: $O(n^3)$ time




Fix two points $p$ and $q$

For any point $r$, we have

$\text{area}(pqr) = \frac{1}{2} b \cdot h$

where

$b = |p.x - q.x|$ and

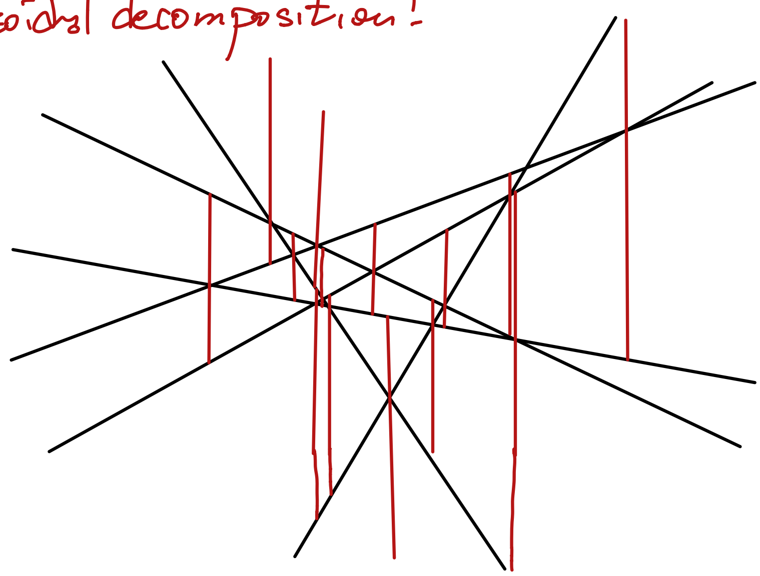
$h =$ vertical distance from $r$ to $\overleftrightarrow{pq}$

So min area triangle $\Delta pq\_$ uses 3rd point closest to $\overleftrightarrow{pq}$

## In the dual:

For each vertex in $\text{arrgh}(P^*)$, we want closest line above or below

Trapezoidal decomposition!



Build $\text{arrgh}(P^*)$
Build trap. decomp
Find shortest vertical edge at each arrangement vertex

$\Big\}$ $O(n^2)$ time

Can we do better? Nobody knows!

Related problem: 3SUM: Given n numbers, do any 3 sum to 0?

Easy $O(n^2)$-time algorithm

For any set $X$, let $\hat{X} = \{(x, x^3) \mid x \in X\}$

Three elements of $X$ sum to 0
iff
Three points in $\hat{X}$ are collinear!

Proof: $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a+b+c)(b-a)(c-b)(c-a)$

Matching $\Omega(n^2)$ lower bound in weak model of computation.

First subquadratic algo for 3SUM: (without bit tricks)

Grønlund Pettie 2014: $O\left(n^2 \frac{\log^2 \log n}{\log n}\right)$ expected

Gold Sharir 2017: $O\left(n^2 \frac{\log\log n}{\log n}\right)$

Chan 2018: $O\left(n^2 \frac{\log^c \log n}{\log^2 n}\right)$

3SUM conjecture: $O(n^{2-\delta})$ time is impossible for all $\delta > 0$ even for integers between $-n^3$ and $n^3$

But no $o(n^2)$-time algorithm for detecting collinear triples of points!