CS 498 TC ✧ Computational Geometry ✧ Spring 2022

Homework 3

Due Tuesday, April 12, 2022 at 8pm

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1. Let P be a convex polygon in the plane with n vertices, represented as a circular doubly-linked list of vertices in counterclockwise order. The following randomized incremental algorithm constructs the Delaunay triangulation of the vertices of P.

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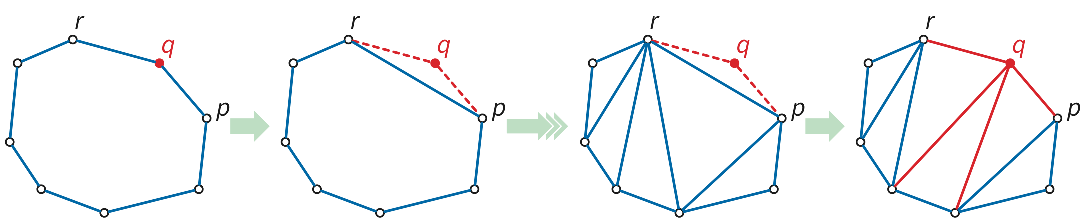
ConvexDelaunay(P):
  if P is a triangle
      return P
  else
      q ← random vertex of P
      p ← pred(q);  r ← succ(q)
      P′ ← P − pq − qr + pr
      T′ ← ConvexDelaunay(P′)
      T ← T′ + pq + qr
      repair T by flipping non-Delaunay edges opposite q
      return T
```



   (a) Prove that the expected number of flips performed by this algorithm is $O(n)$.

   (b) We already know from Lawson's algorithm that we can perform the necessary flips in $O(1)$ time each. But how do we choose the random vertex q in $O(1)$ time?

2. Let P be any set of points in the plane. A triangulation of P is a planar straight-line graph whose vertices are precisely the points P, and whose bounded faces are triangles whose union is the convex hull of P.

   There are several different ways to measure the *quality* of a triangulation. The goal of this question is to prove that the Delaunay triangulation of P is the best possible triangulation of P, for a few different definitions of "best".

   For example, Chapter 9.1 of "Four Marks" contains a proof that among all triangulations of P, the Delaunay triangulation of P has the largest minimum angle. (See also Lecture 11 in David Mount's lecture notes, and Chapter 5.6 of the Swiss lecture notes.)

   (a) Let T be any triangulation of P. Prove that if every interior angle of T is acute (that is, strictly smaller than a right angle), then T is the Delaunay triangulation of P.

(b) For any triangle $\triangle$ with vertices pqr, define

$$Vol(\triangle) = area(\triangle) \cdot (\|p\|^2 + \|q\|^2 + \|r\|^2)$$

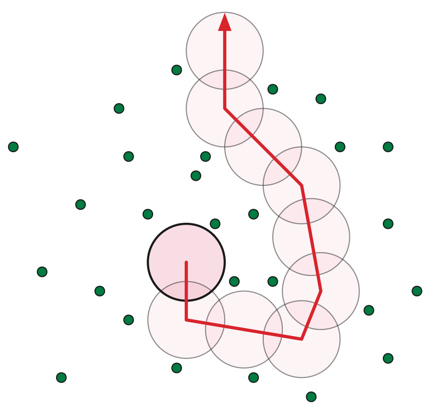
where $\|(a, b)\|^2 = a^2 + b^2$ is the squared Euclidean norm. For any triangulation T, let $Vol(T) = \sum_{\triangle \in T} Vol(\triangle)$. Prove that among all triangulations of P, the Delaunay triangulation of P minimizes $Vol(T)$. *[Hint: Why is the function called "Vol"?]*

*(c) For any triangle $\triangle$ with side lengths a, b, and c, define

$$\Phi(\triangle) = area(\triangle) \cdot (a^2 + b^2 + c^2).$$

For any triangulation T, let $\Phi(T) = \sum_{\triangle \in T} \Phi(\triangle)$. Prove that among all triangulations of P, the Delaunay triangulation of P minimizes $\Phi(T)$.

3. Suppose you want to get a balloon out of a forest. The forest is modeled as a set P of point sin the plane, each representing the location of a tree; the balloon is modeled as a circle, specified by its center c and radius r. Your goal is to move the balloon arbitrarily far away from the trees by continuous translation, without touching any of the trees.



(a) Describe an algorithm that either computes an *escape path* for the balloon or correctly reports that no such path exists. An escape path is a path π from the initial center c to some point far outside the convex hull of P, such that every point on π has distance greater than r to every point in P.

(b) Suppose you don't know the size or the initial location of the balloon. Describe an algorithm to preprocess the point P into a data structure that supports queries of the following form: Given a center point c, what is the maximum radius r of a balloon centered at c that can escape the forest?

4. Suppose you are given a set of n halfplanes that define a convex polygon P in the plane. Show that each of the following objects inside P can be found in $O(n)$ expected time using linear programming. ("Axis-aligned" means all are edges either horizontal or vertical.)

   (a) The largest axis-aligned square inside P.

   (b) The axis-aligned rectangle inside P with maximum perimeter.

   (c) Two interior-disjoint axis-aligned squares inside P with maximum total perimeter.

   (d) The largest circle inside P.

5. Suppose you are given a set of n red points and n blue points in the plane.

   (a) Give an example where the smallest circle containing the red points also contains a blue point, but some other circle has all red points inside and all blue points outside.

   (b) Describe and analyze an algorithm to determine if there is a circle with all red points inside and all blue points outside.

   (c) Describe and analyze an algorithm to determine if there are two *disjoint* circles, one containing all the red points but no blue points, and the other containing all the blue points but no red points. *[Hint: This is easier than it looks.]*