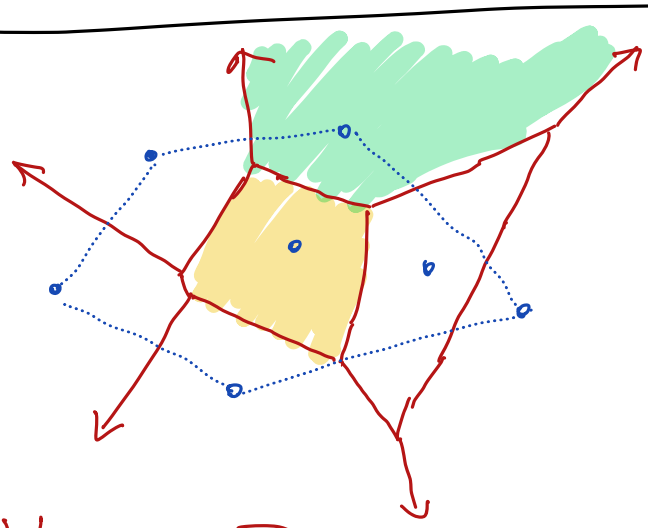
Hw 1 graded (mostly) — Hw2 solutions out tonight

Hw 2 can be submitted — Midterm available Sun am – Tue 8pm – 2 hours



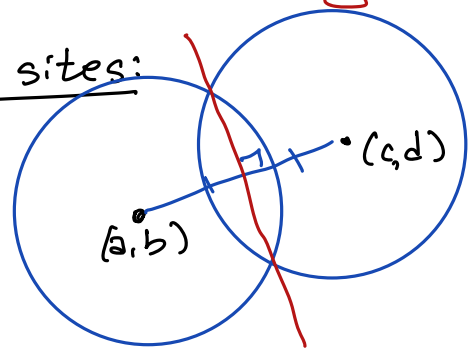
Voronoi Diagram

Given set P of points called "sites"

Voronoi diagram = partition into regions:

$Vor(p) = \{ q \mid p \text{ is closest site to } q \}$

2 sites:



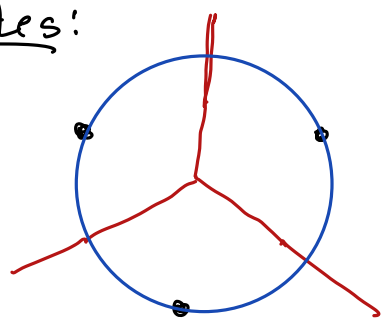
perpendicular bisector

$$\|(a,b)-(x,y)\|^2 = \|(c,d)-(x,y)\|^2$$

$$(a-x)^2+(b-y)^2 = (c-x)^2+(d-y)^2$$

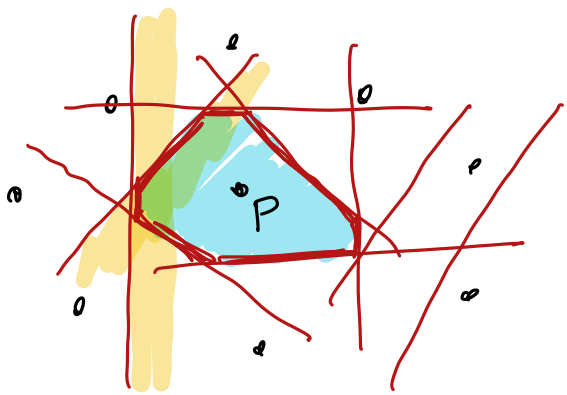
$$(a-c)x+(b-d)y = \tfrac{1}{2}(a^2+b^2)-\tfrac{1}{2}(c^2+d^2)$$

3 sites:



3 bisectors meet at the center of the circumcircle of the 3 sites

More than 3 sites:



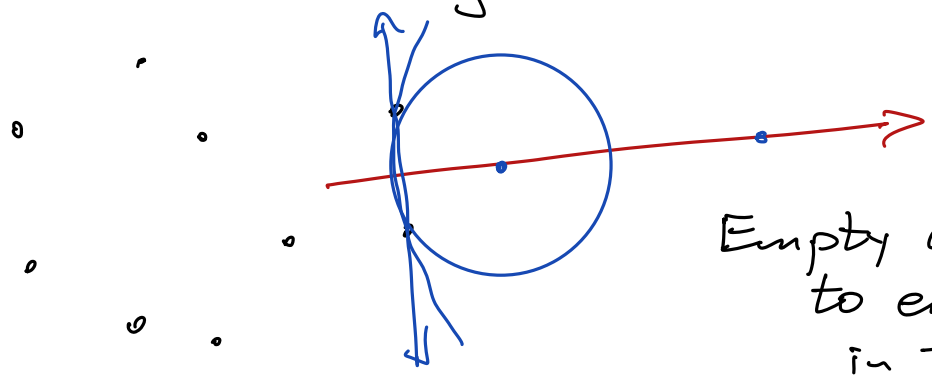
Vor(p) is intersection of n-1 halfplanes defined by perp. bisectors.

→ convex polygon

General position → no 4 pts are cocircular
→ every Vor. vertex has deg 3.

Vor(p) is unbounded iff p is vertex of conv(P)

Unbounded Vor. edge $\perp$ convex hull edge



Empty circles degenerate to empty halfplanes in the limit.

PSLG Euler's formula

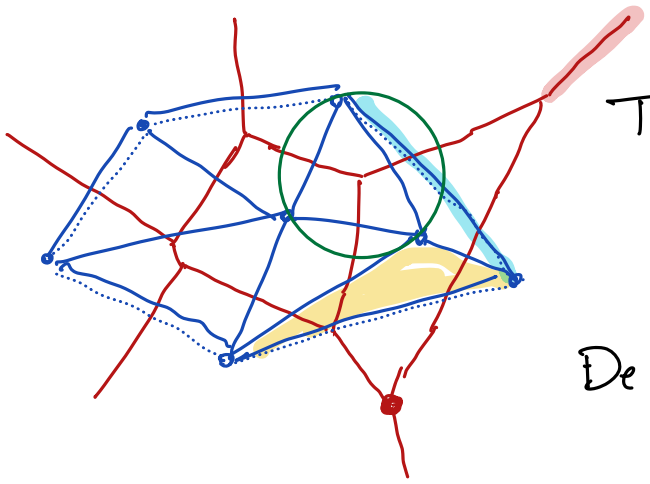
$\Rightarrow$ Vor diagram has $2n-2-h$ vertices (not counting ∞)

and $3n-3-h$ edges

(assuming general position)

otherwise $\leq$

Dual of Vor diagram = Delaunay triangulation (Delone)

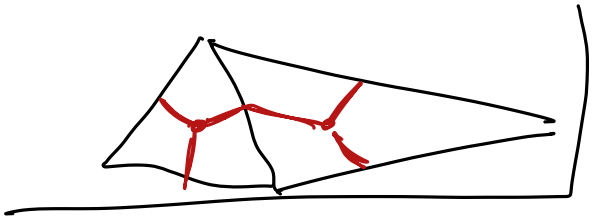


Three point pqr define a Delaunay triangle iff circ(pqr) is empty

Delaunay edge pq $\Leftrightarrow$ $\exists$ circle with pq inside other sites outside

PSLG $2n-2-h$ faces
$3n-3-h$ edges where h = # convex hull edges

Dual means <u>same data structure</u>



Circles:

circle with center (a,b) and radius r:

$$(x-a)^2+(y-b)^2=r^2$$

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

$$x^2+y^2-2\alpha x-2\beta y+\gamma=0$$

$$\Downarrow$$

circle center (α,β)

radius $\sqrt{\gamma-(\alpha^2+\beta^2)}$

Circumcircle of (a,b) (c,d) (e,f):

$$\det\begin{bmatrix}1 & a & b & a^2+b^2\\ 1 & c & d & c^2+d^2\\ 1 & e & f & e^2+f^2\\ 1 & x & y & x^2+y^2\end{bmatrix}=0$$

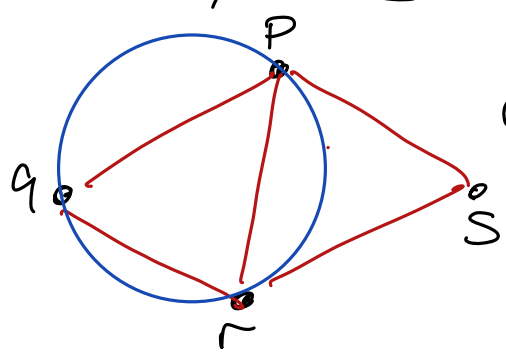
line thru (a,b) (c,d):

$$\det\begin{bmatrix}1 & a & b\\ 1 & c & d\\ 1 & x & y\end{bmatrix}=0$$

$$\| $$

$$\underbrace{(x^2+y^2)\begin{vmatrix}1 & a & b\\ 1 & c & d\\ 1 & e & f\end{vmatrix}}_{\text{collinear} \to \text{ circle degenerates to a line}}-y\begin{vmatrix}1 & a & a^2+b^2\\ 1 & c & c^2+d^2\\ 1 & e & e^2+f^2\end{vmatrix}+x\begin{vmatrix}1 & b & a^2+b^2\\ 1 & d & c^2+d^2\\ 1 & f & e^2+f^2\end{vmatrix}-1\begin{vmatrix}a & b & a^2+b^2\\ c & d & c^2+d^2\\ e & f & e^2+f^2\end{vmatrix}$$

Delaunay triangulation of 4 points

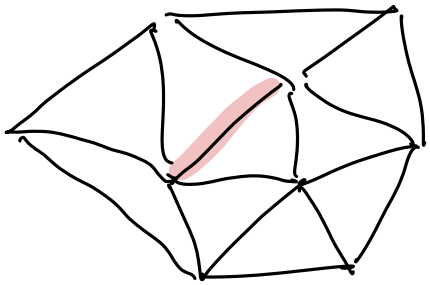


equivalent!
- s is outside circ(pqr)
- q is outside circ(prs)
- p is inside circ(qrs)
- r is inside circ(pqs)

$$-\det\begin{bmatrix}1 & p_x & p_y & p_x^2+p_y^2\\ 1 & q_x & q_y & \text{—}\\ 1 & r_x & r_y & \text{—}\\ 1 & s_x & s_y & \text{—}\end{bmatrix}>0$$

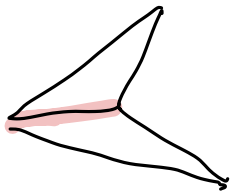
(assuming pqr ccw)

More generally a triangulation is <u>locally</u> Delaunay if every interior edge passes this test



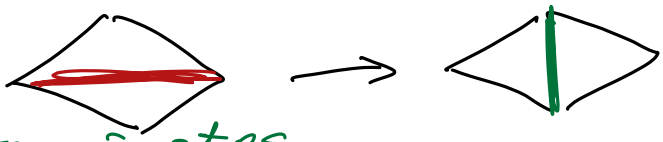
Theorem:

locally Delaunay = Delaunay



Lawson's flip algorithm

1. Triangulate P however
2. while there is a locally non-Del edge
   Flip it



Theorem: This algo terminates after $O(n^2)$ flips $\rightarrow$ Delaunay Δn!