

Admin

Convolution for Minkowski sums

Fréchet distance(s)

→ two more lectures

HW4? (short)

Final exam?

Minkowski Sum  $A+B = \{a+b \mid a \in A \text{ and } b \in B\}$

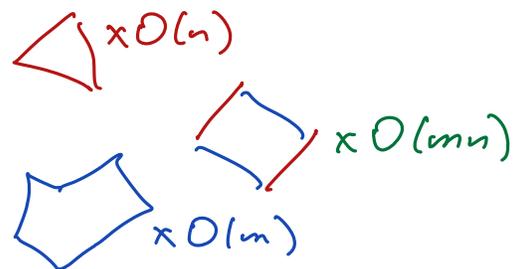
motion planning / collision detection  
via configuration space

• A and B are convex — merge edges  $O(n+m)$  time  
 $\Rightarrow A+B$  also convex

• A and B are nonconvex

$\rightarrow A+B = \text{int}A + \text{vert}B$   
 $\cup \text{edge}A + \text{edge}B$   
 $\cup \text{vert}A + \text{int}B$

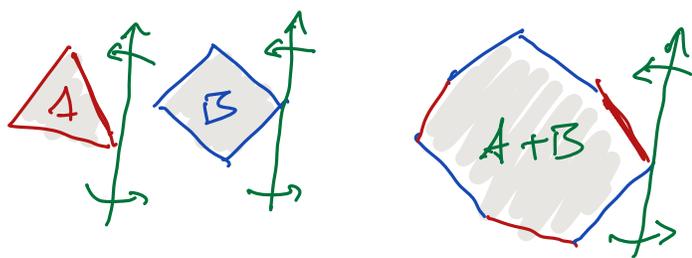
$O(m^2n^2)$  time



$\rightarrow$  decompose  $A+B$  into convex pieces

$$A+B = \bigcup_{i,j} (A_i + B_j)$$

$\rightarrow$  convolution  $A * B$

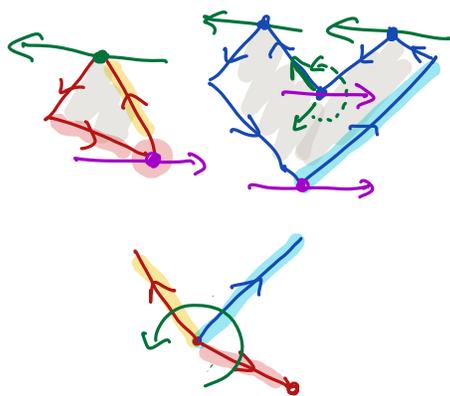


Suppose

$$A = a_1 a_2 \dots a_m$$

$$B = b_1 b_2 \dots b_n$$

are simple polygons



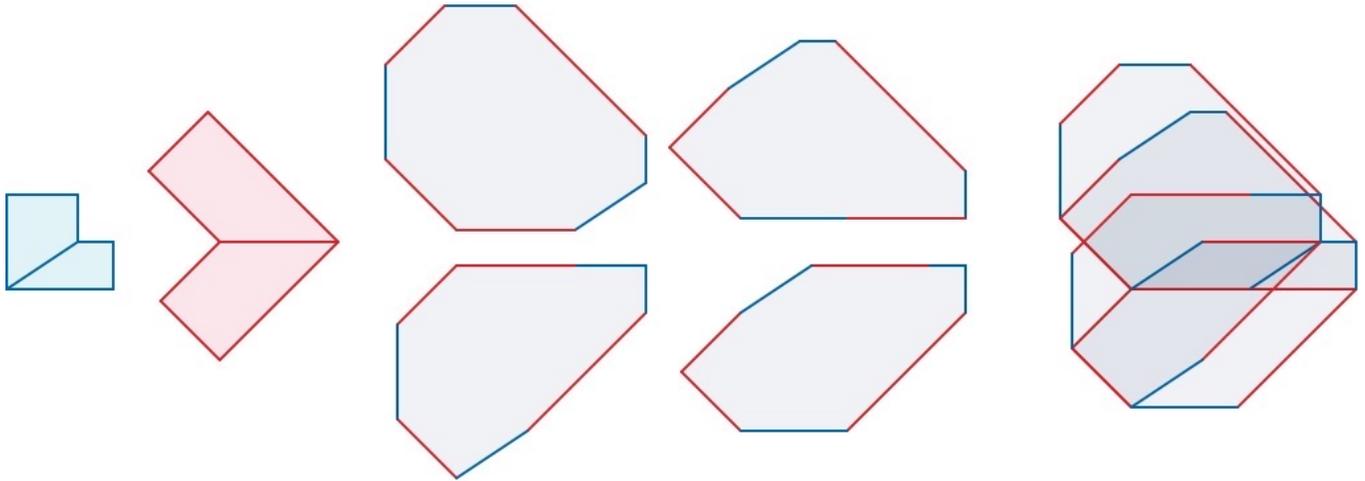
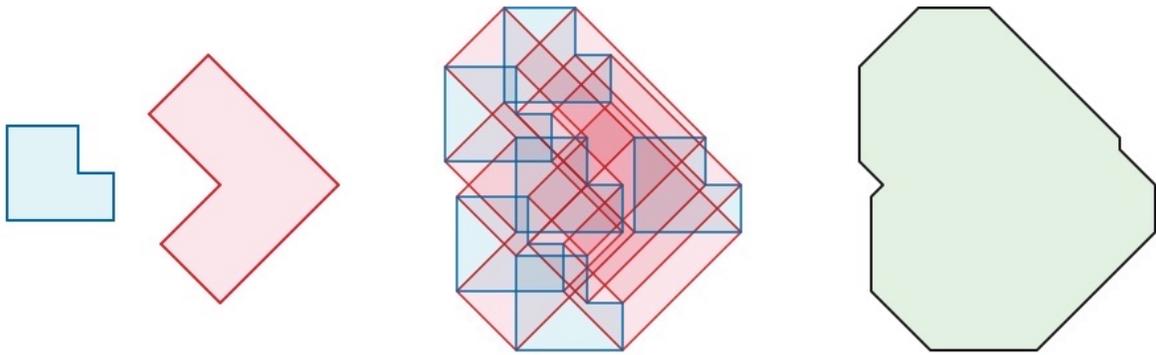
The convolution  $A * B$  is a directed graph with two types of edges:

•  $(a_i \rightarrow a_{i+1}) + b_j$  for all  $i, j$  such that  
 $b_{j-1} \rightarrow b_j, a_i \rightarrow a_{i+1}, b_j \rightarrow b_{j+1}$  ccw order

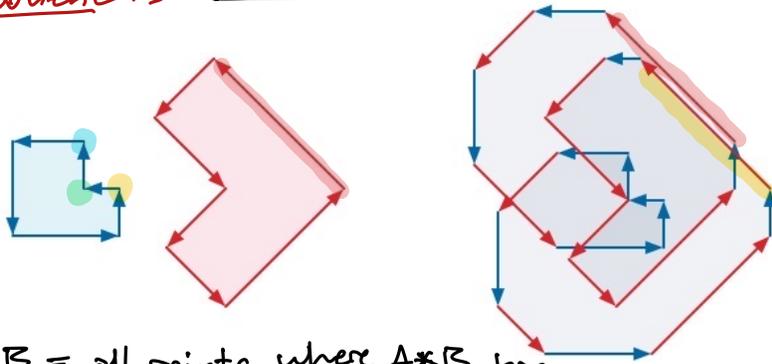
•  $a_i + (b_j \rightarrow b_{j+1})$  such that

$a_{i-1} \rightarrow a_i, b_j \rightarrow b_{j-1}, a_i \rightarrow a_{i+1}$  ccw order

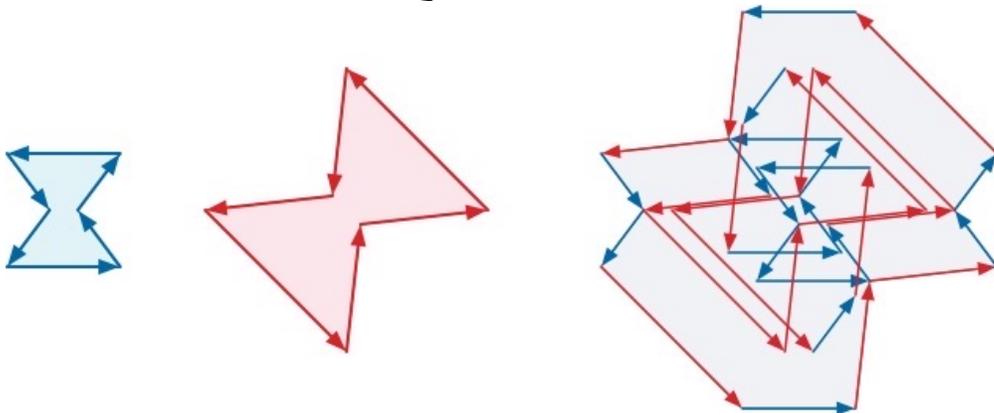
$O(mn)$  edges



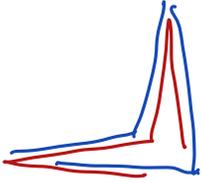
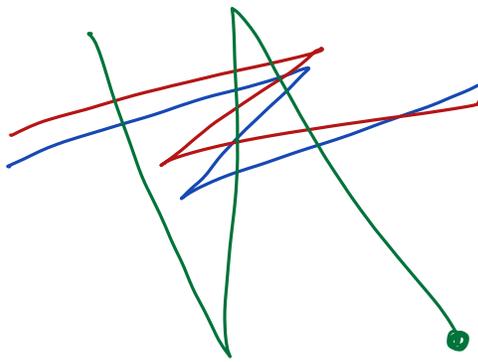
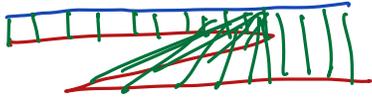
$A * B$  is directed straight-line graph  
 indegree = outdegree not necessarily connected!  
 each component is Eulerian



$A \oplus B$  = all points where  $A * B$  has  
 positive winding #  
 $O(n+m+k)$  time [Guibas Seidel]



# Fréchet distance

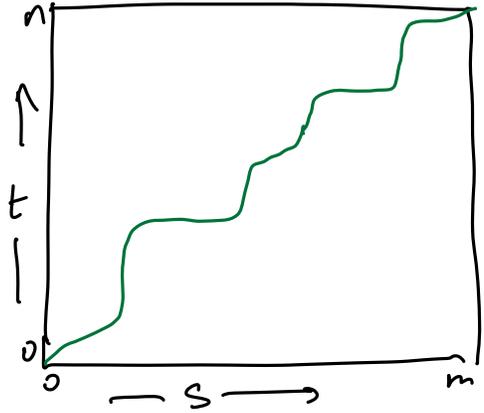


curves  
 $P: [0, m] \rightarrow \mathbb{R}^2$   
 $Q: [0, n] \rightarrow \mathbb{R}^2$

$$\Delta(s, t) = \text{dist}(P(s), Q(t))$$

human walks along  $P$  / without backing up  
 dog walks along  $Q$

Fréchet( $P, Q$ ) = min length of a leash



monotone path through parameter space  $[0, m] \times [0, n]$

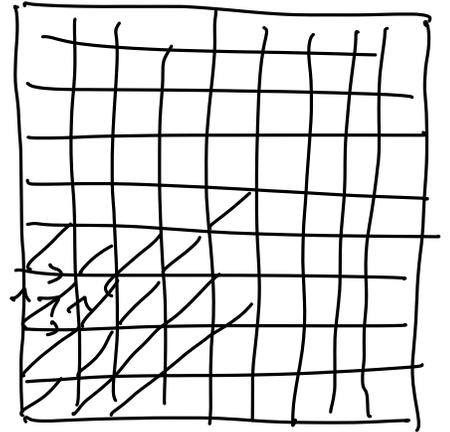
$$\text{Fréchet} = \min_{\Pi} \max_{(s,t) \in \Pi} \Delta(s,t)$$

## Start: Discrete Fréchet distance

At each step, dog and/or human move along one edge

minimize max vertex-to-vertex distance

$\Leftrightarrow$   
 min max path in parameter dog  $\rightarrow$   
 from  $(0,0)$  to  $(m,n)$



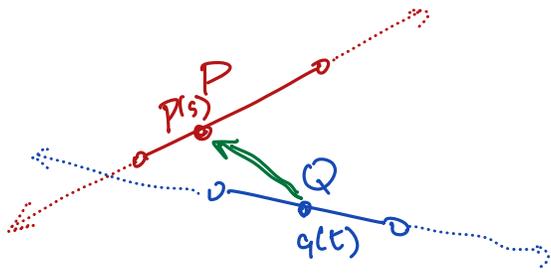
$$DF(i, j) = \max(\Delta(i, j), \min(DF(i+1, j), DF(i, j+1), DF(i+1, j+1)))$$

$O(mn)$  time via dyn. prog.



But distance now depends on discretization of the paths, not just their geometry.

Is Fréchet  $(P, Q) \leq \delta$ ?



First consider special case where P and Q are segments

Lemma: Free space is convex!

intersection of  $\square$  with interior of ellipse

$$P: \mathbb{R} \rightarrow \mathbb{R}^2$$

$$Q: \mathbb{R} \rightarrow \mathbb{R}^2$$

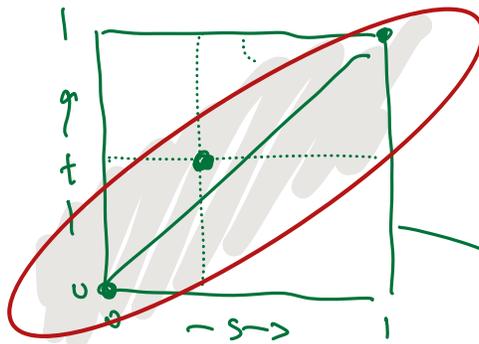
} affine = linear + translation

$$\Delta(s, t) = P(s) - Q(t)$$

$$\Delta: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ affine}$$

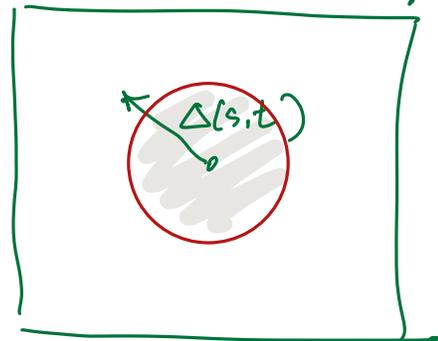
$(s, t)$  is free iff  $\|\Delta(s, t)\| \leq \delta$

preimage  $\Delta^{-1}(\delta\text{-ball})$  is elliptical disk  $\square$



$(s, t)$  is free if  $\text{dist}(P(s), Q(t)) \leq \delta$

Is there a free monotone path from  $(0, 0)$  to  $(1, 1)$ ?



More generally, configuration space is  $m \times n$  grid  
Free space is convex inside each cell

