You may use whatever sources you like—paper, electronic, human, or other—but please cite every source that you use—paper, electronic, human, or other. Please email your solutions to jeffe@cs.illinois.edu as a PDF file. Groups of up to three students may submit joint solutions.

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1. Fun with Euler's formula! Most of these are easy to look up, but try to solve them yourself. ***By convention, all planar maps are connected.***

   (a) Prove that every simple *bipartite* planar graph has at most $2n - 4$ edges, ***except the simple graph with only one edge***.

   (b) Prove that every planar map has either a vertex with degree at most 3 or a face with degree at most 3.

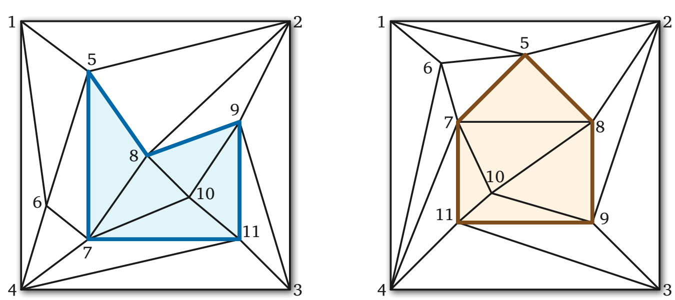
   (c) A planar map is ***regular*** if all faces have the same degree and all vertices have the same degree. (More formally, a surface map is regular if it has a flag-transitive automorphism group, but for planar maps, the simpler condition is equivalent.) Give a complete list of all regular planar maps, and prove your list is complete. *[Hint: The list is infinite!]*

   (d) A simple bipartite planar map is ***bibipartite*** if its dual map is also simple and bipartite. Give a complete list of all bibipartite planar maps, and prove your list is complete. *[Hint: The list is non-empty!]*

   (e) Suppose we arbitrarily color each edge of a simple planar map either red or blue. Prove that there is a vertex whose incident red edges are consecutive.

   (f) Prove that there are constants α and β such that every simple planar graph with n vertices has an independent subset of at least n/α vertices, each with degree less than β. (Biedl and Wilkinson [2] describe the best known tradeoffs between α and β, but you should be able prove $\alpha \le 12$ and $\beta \le 12$ directly from Euler's formula.)

2. Let P be a simple polygon inside a large rectangle R. We say that a (straight-line) triangulation of R ***conforms*** to P if P is the union of triangles in the triangulation. (A conforming triangulation can have vertices on the edges or even in the interior of P.) We say that two triangulations of R are ***isomorphic*** if there is a bijection between the vertices, edges, and faces of one triangulation and the vertices, edges, and faces of the other.



Isomorphic triangulations conforming to two simple polygons.

   For any simple polygons P and Q, let $\boldsymbol{\Delta(P,Q)}$ denote the minimum possible number of vertices in isomorphic triangulations of some common bounding rectangle that conform to P and Q. Our proof of the Schönflies theorem for simple polygons implies that $\Delta(P,Q)$ is always *finite*, but the triangulations implied by the proof may have exponential complexity. (You should confirm this!)

(a) Prove that if P is a simple n-gon and Q is a triangle, then $\Delta(P,Q) = O(n)$.

(b) Prove that if P and Q are both simple n-gons, then $\Delta(P,Q) = O(n^2)$. *[Hint: Use part (a)!]*

*(c) For any integer $n \geq 3$, describe two simple n-gons P and Q such that $\Delta(P,Q) = \Omega(n^2)$.

Aronov *et al.* [1] consider a similar problem, where only the interiors of the polygons need to be triangulated. It is an open question whether $\Delta(P,Q)$ can be computed, or even approximated up to constant factors, in polynomial time.

3. The funnel algorithm for computing shortest homotopic paths can be modified to compute shortest (freely) homotopic *cycles* as well [3].

   (a) Let P be a triangulated polygon with holes in the plane, and let γ be a ***noncontractible*** closed polygonal chain in the interior of P. Prove that the shortest cycle in P that is freely homotopic to γ passes through a vertex of P.

   (b) *Sketch* an algorithm to compute the shortest cycle in a given polygon with holes that is freely homotopic to a given polygonal cycle. Don't develop the algorithm from scratch; just describe the necessary changes to the funnel algorithm.

   (c) Now consider an arbitrary ***band*** B constructed by gluing a sequence of Euclidean triangles into an annulus. Sketch an algorithm to find the shortest cycle that winds once around the band. The shortest cycle might not pass through a vertex of B; your algorithm should handle this case correctly.

   (d) What if B is a Möbius band instead of an annulus?

References

[1] Boris Aronov, Raimund Seidel, and Diane Souvaine. On compatible triangulations of simple polygons. *Comput. Geom. Theory Appl.* 3(1):27–35, 1993.

[2] Therese Biedl and Dana F. Wilkinson. Bounded-degree independent sets in planar graphs. *Theory Comput. Systems* 38(3):253–278, 2005.

[3] John Hershberger and Jack Snoeyink. Computing minimum length paths of a given homotopy class. *Comput. Geom. Theory Appl.* 4:63–98, 1994.

[4] David G. Kirkpatrick. Optimal search in planar subdivisions. *SIAM J. Comput.* 12(1):28–35, 1983.