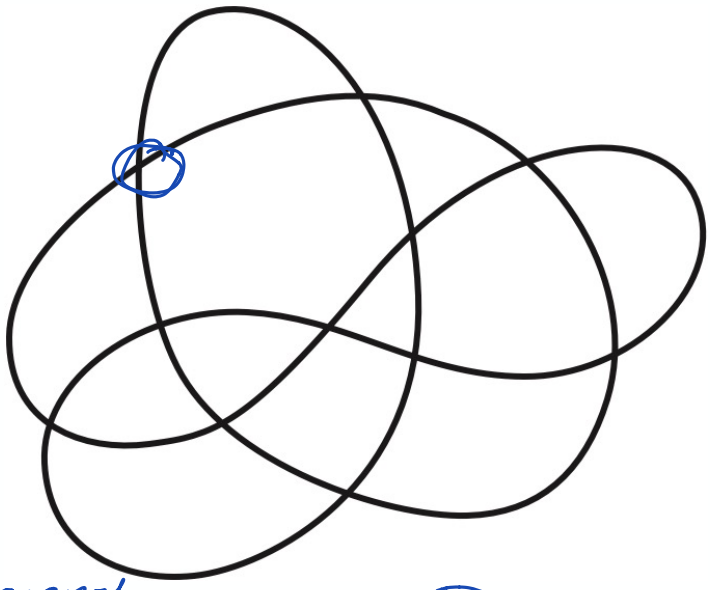
Generic closed curves



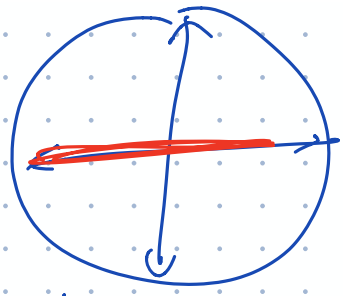
Closed curve $\gamma : S^1 \to \mathbb{R}^2$

self-intersection $\gamma(t) = \gamma(t')$

transverse: For small enough ε

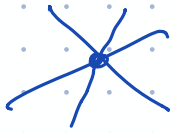
subcurves $\gamma(t-\varepsilon, t+\varepsilon)$
$\gamma(t'-\varepsilon, t'+\varepsilon)$

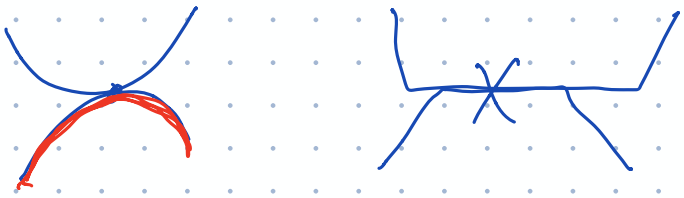
are homeomorphic to two orthogonal lines



A curve is generic iff

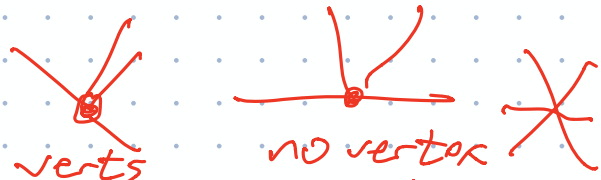
- Every self-int is transverse
- Every self-int is pairwise
  no $\gamma(t) = \gamma(t') = \gamma(t'')$





self-intersections is finite

Any curve approximated by polygon
$\downarrow$ perturb
generic polygon
$\|$
generic closed curve



verts distinct

no vertex on edge

no 3-way ints

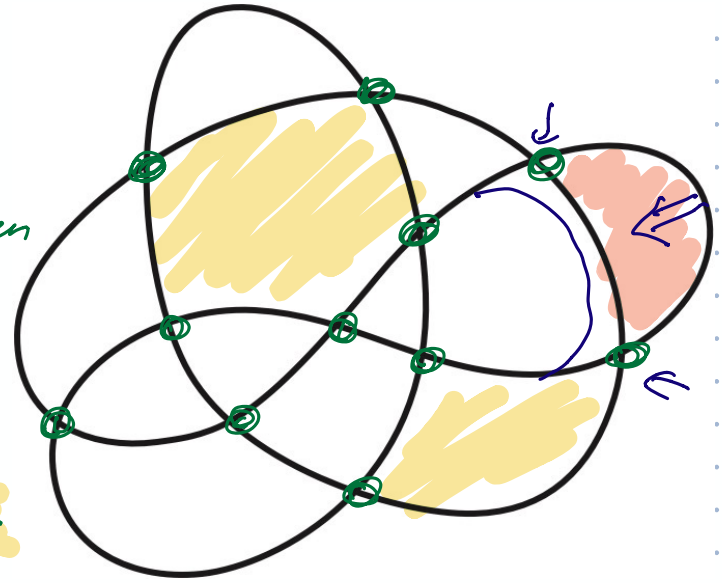
Two curves are isotopic if they are homotopic through generic curves with same # of self-intersections

Equiv (in $\mathbb{R}^2$) orientation-preserving homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\gamma = H \circ \gamma'$

Image graph

vertices = self-ints

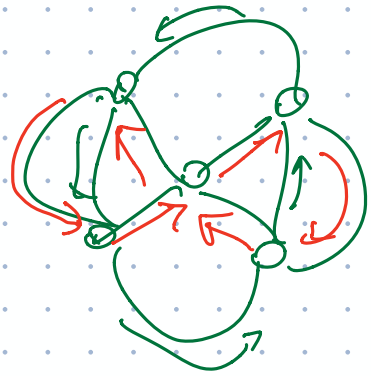
edges = subcurves between vertices



Almost always connected 4-regular plane graph.

Simple →  no vertices!

But not every conn. 4-reg. planar graph is image of a curve



Every curve is an Euler tour of its image

Gaussian Euler tour

After entering any node leave thru opposite edge

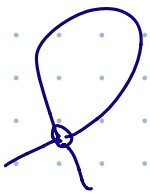
image of multicurve

$\gamma : S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^2$

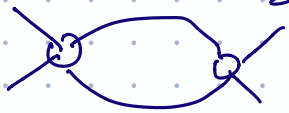
Faces of a curve = faces of image graph

= components of $\mathbb{R}^2 \setminus \operatorname{im}\gamma$

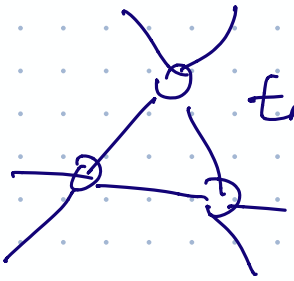
= open disks except outer face (comp. of closed disk)



monogon

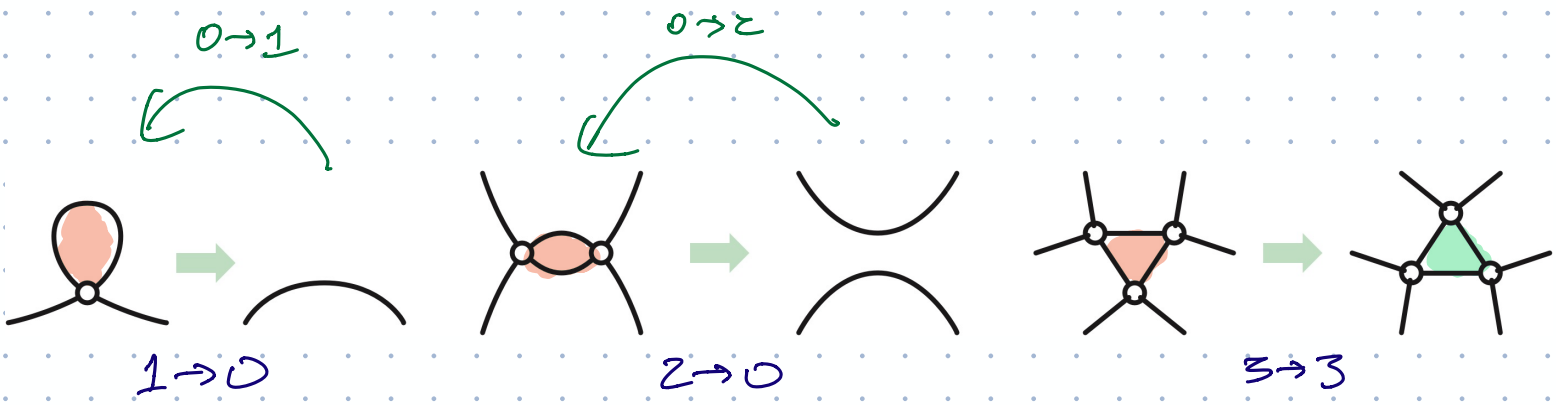


bigon



triangle

Homotopy moves: Change a small neighborhood of one face with ≤ 3 vertices.



Every homotopy between polygons $\longrightarrow$ vertex moves

$\Downarrow$

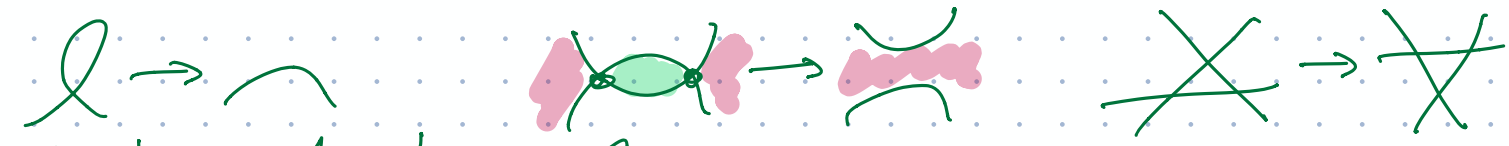
Every homotopy between generic curves

$\hookrightarrow$ finite sequence of homotopy moves.

Theorem: Every generic planar curve with n vertices has $n+2$ faces.

(Euler's formula) $V-E+F=2$, $E=2V$

Proof: Fix $\gamma \to \gamma_1 \to \gamma_2 \to \gamma_3 \to \cdots \to \gamma_k = 0$



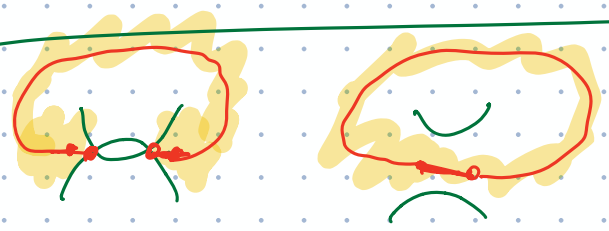
n verts, f faces $\longrightarrow$ $n-1$ verts, $f-1$ faces

n, f $\longrightarrow$ $n-2$, $f-2$

$\hat{f}$ $\longrightarrow$ $\hat{f}$

$f-n$ is preserved

$n=0$, $f=2$ $\square$



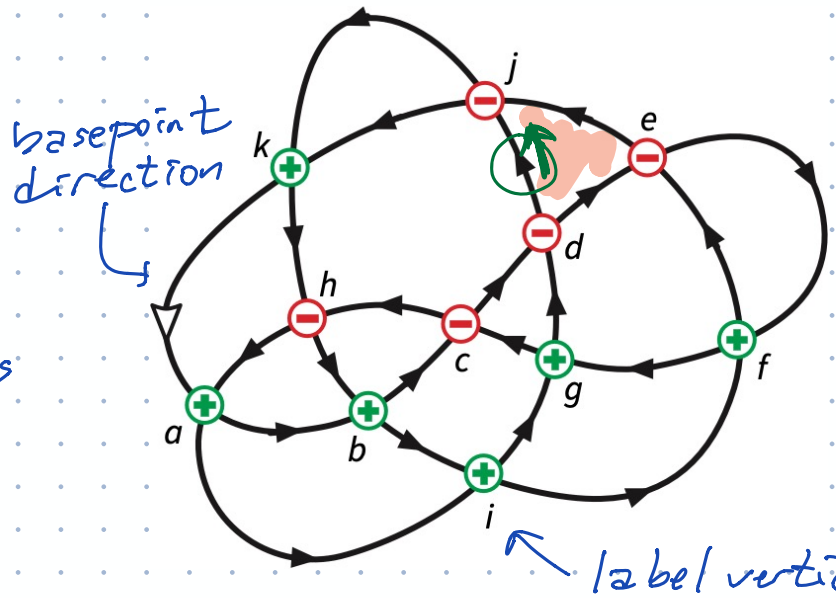
$2\to 0$ move either merges two faces or disconnects the image graph.

impossible

knot diagram

Gauss code

sequence of vertex labels along curve

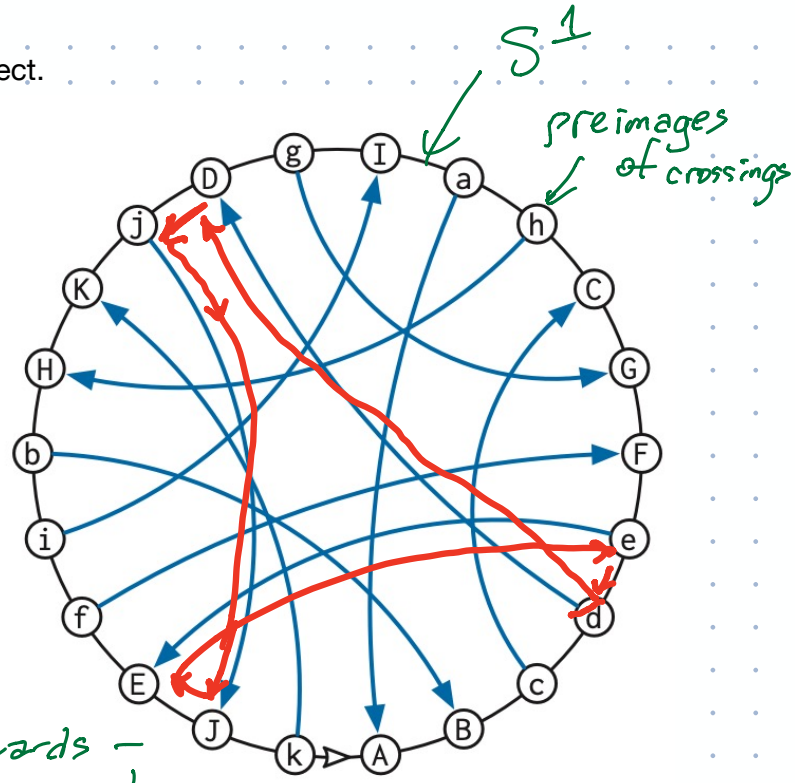


signed Gauss code:

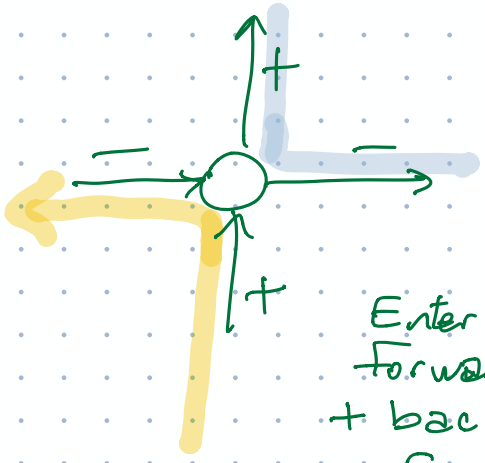
ABcdeFGChaIgD....

Note added after lecture: This is the correct convention.
The vertex signs and Gauss code above right are also correct.

Gauss diagram:



Tracing faces



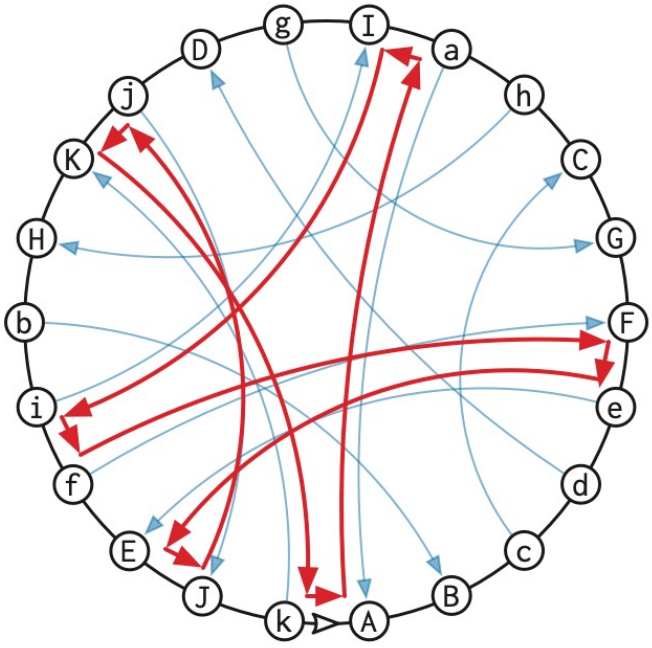
Enter	Leave
+ forward	− backwards
+ back	− forward
− forward	+ forward
− back	+ back

ABab

Note added after lecture: These instructions are for tracing a face counterclockwise (on the left of a moving point), but the two examples in the Gauss diagrams are tracing the face clockwise (on the right).

in $O(n)$ time extract all $n+2$ faces.

Signed Gauss code is consistent with a planar curve IFF $f=n+2$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