Planar graphs + maps

"a graph we can draw in the plane without crossing edges."

(V, E) $\quad V =$ set "vertices"

~~E = set of 2-element subsets of V~~

~~"edges"~~

$(V, D, rev, head)$ $\quad V =$ set "vertices" $\neq \emptyset$

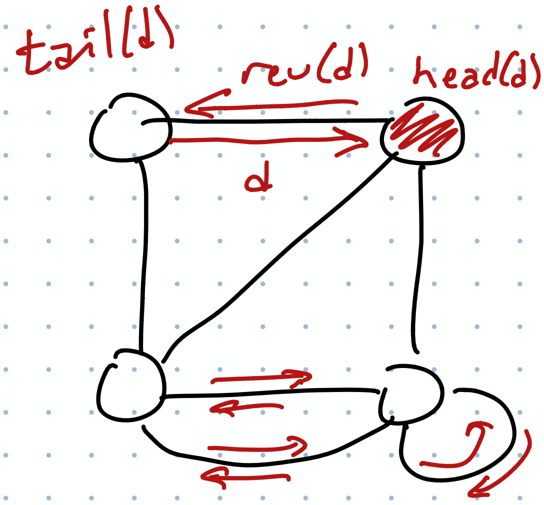
half-edge
flags
arcs
brins

$\longrightarrow$ $D =$ set "darts"

$rev : D \to D$ permutation

$rev(rev(d)) = d \neq rev(d)$

$head : D \to V$



$edge = \{d, rev(d)\}$

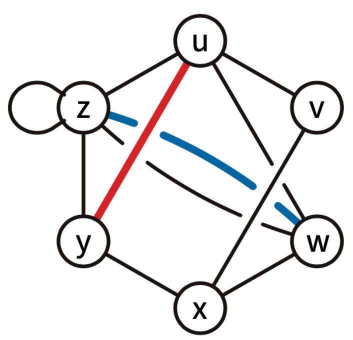
$e = \{e^+, e^-\}$

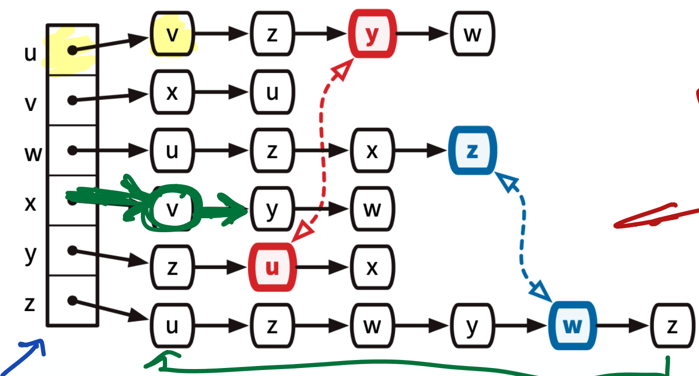
$tail(d) = head(rev(d))$

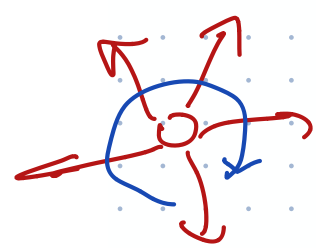
Incidence list

list of outgoing darts

each d record head(d) and rev(d)







indexed by vertices

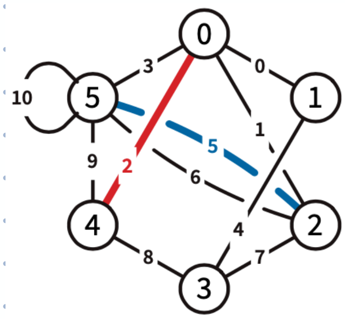
random access

iterate

Index vertices $0..n-1$
edges $0..m-1$
darts $0..2m-1$

edge $i = \{\text{dart } 2i, \text{dart } 2i+1\}$

$\text{rev}(d) = d \,\hat{}\, 1$



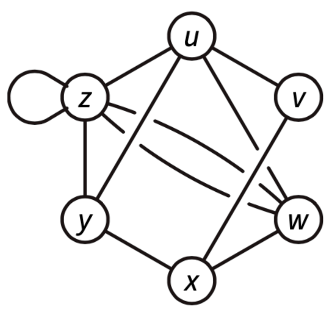
first[v] = any dart leaving v

first	0	1	10	9	3	20																
head	1	0	2	0	4	0	3	0	3	1	5	2	5	2	3	2	4	3	5	4	5	5
next	2	8	4	10	6	17	0	11	1	15	12	13	14	19	2	16	9	18	5	20	21	7

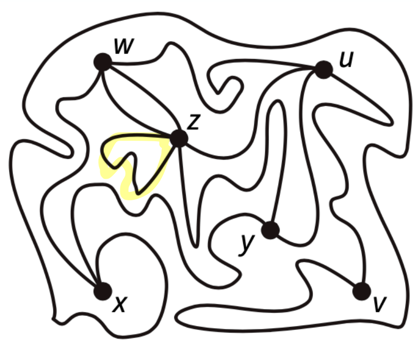
permutation

Convenient: prev[0..2m-1]

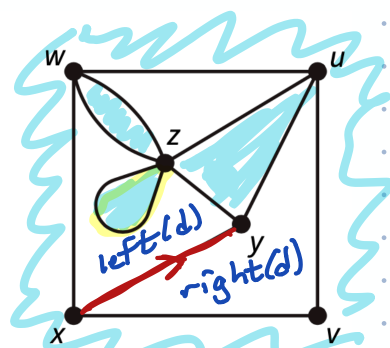
Graphs are not necessarily simple!



planar graph



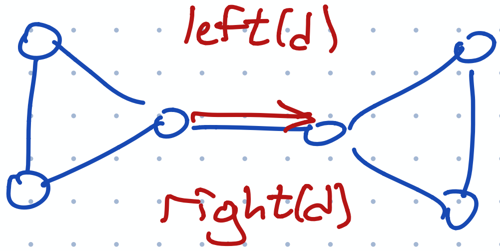
planar embedding
"plane graph"



planar embedding

V → points   E → interior-disjoint, simple paths
↳ piecewise linear

Faces = components of complement of image



– bridge –

Shores of a dart/edge

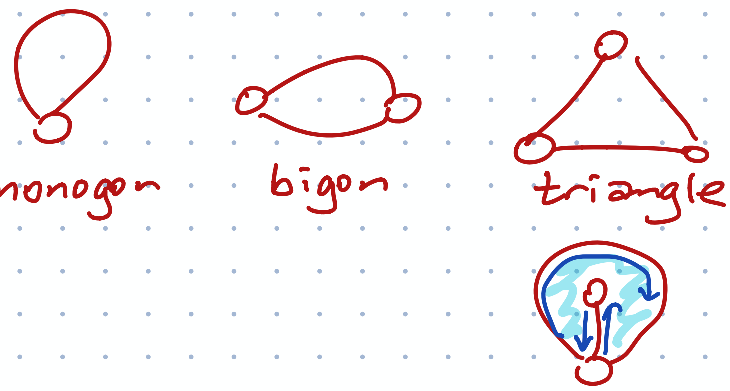
left(d) = right(rev(d))

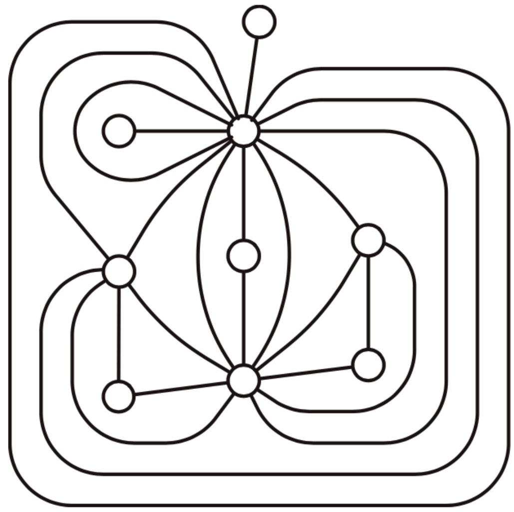
right(d) = left(rev(d))

$\deg(v) = \#\{d \mid tail(d) = v\}$

$\deg(f) = \#\{d \mid right(d) = f\}$

$\boxed{\text{Triangulation} \Leftrightarrow \deg(f) = 3}$





triangulation

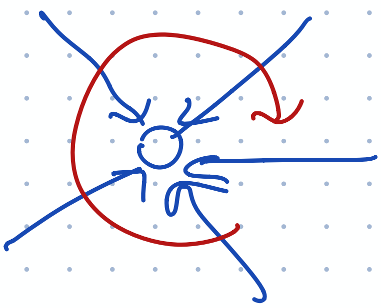
Δ-complex

Planar map = (V, E, F)

head, tail

right, left

Rotation system



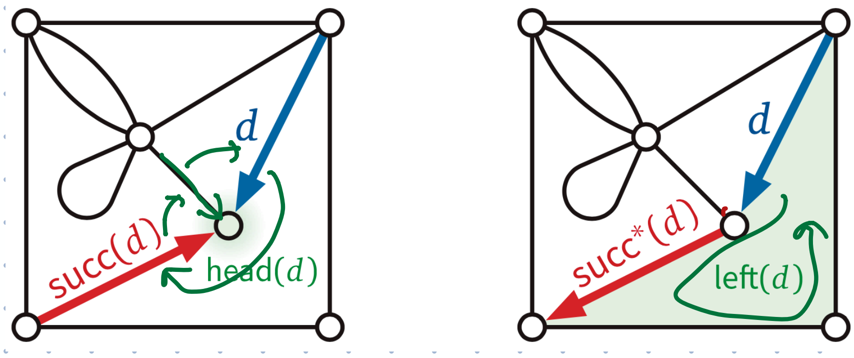
succ : D → D

permutation

succ(d) = next after d

in clockwise order around head(d)

$succ(d) = rev(next(rev(d)))$



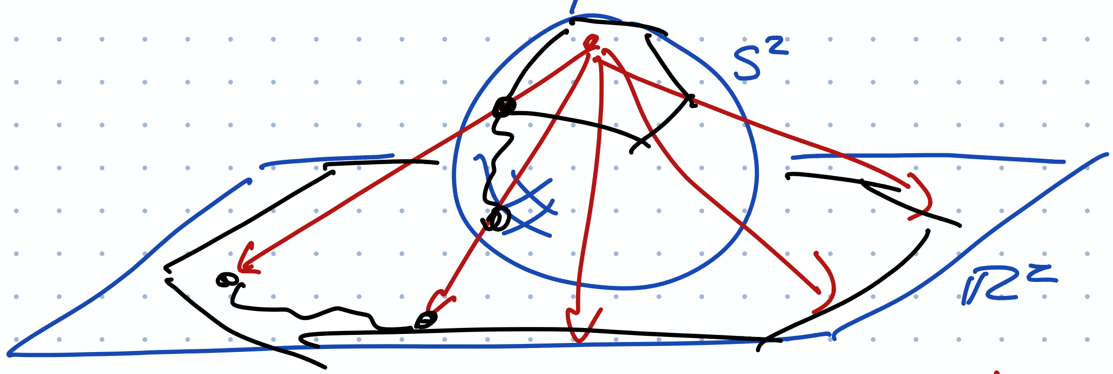
cycles of succ = vertices
orbits

$succ^*(d) = rev(succ(d))$

= next after d in counterclockwise order around left(d)

(D, rev, succ) ← combinatorial map
rotation system

Two maps on sphere with same rot system are homeomorphic (almost)



stereographic projection

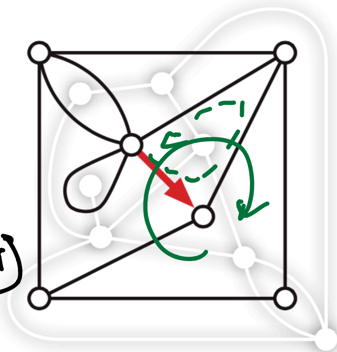
Two planar maps with same rot system and same outer face are homeomorphic

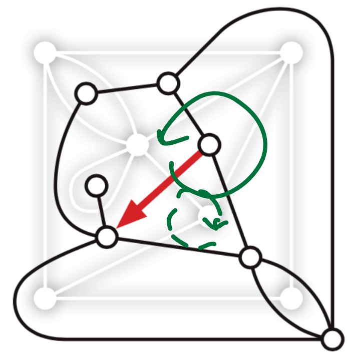
Duality

Given planar map

$\Sigma = (V, E, F)$

Dual map $\Sigma^* = (F^*, E^*, V^*)$





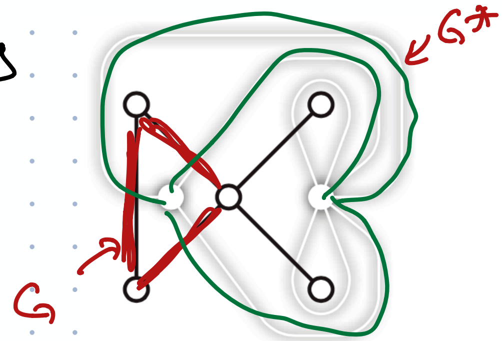
F^* = one point in each face

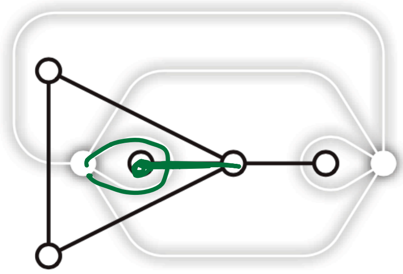
E^* = path btwn dual vertices

$\Downarrow$ e^* crosses e transversely once
no other edge in E

V^* = regions containing one vertex in V

$(\Sigma^*)^* \cong \Sigma$





$\Sigma = (D, \mathrm{rev}, \mathrm{succ})$ $\Sigma^* = (D, \mathrm{rev}, \mathrm{rev}(\mathrm{succ}))$

succ*

V = orbits of succ V^* = orbits of succ*

E = orbits of rev E^* = orbits of rev

F = orbits of rev(succ) F^* = orbits of rev(succ*)

Duality is not a "transformation"

It's a type cast

The same data represents both Σ and Σ^*

Primal Σ	Dual Σ^*	Primal Σ	Dual Σ^*
vertex v	face v^*	head(d)	left(d^*)
dart d	dart d^*	tail(d)	right(d^*)
edge e	edge e^*	left(d)	head(d^*)
face f	vertex f^*	right(d)	tail(d^*)
succ	rev ∘ succ	clockwise	counterclockwise
rev	rev