Generic closed curve

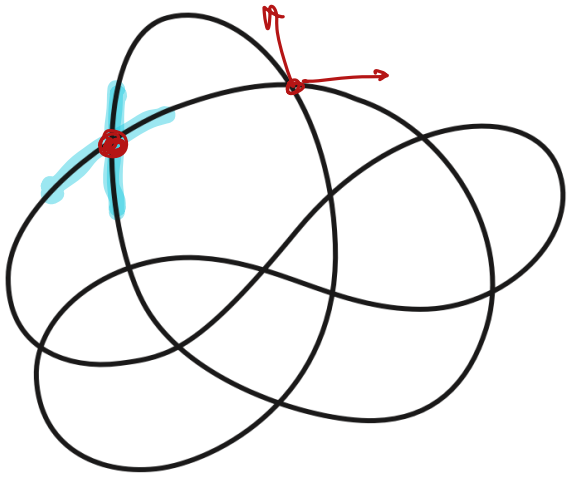
$\gamma : S^1 \to \mathbb{R}^2$

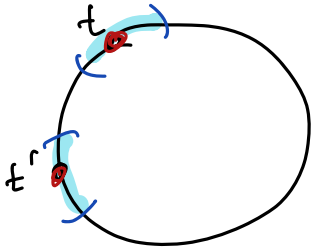
simple (injective) is boring

self-intersections: $\gamma(t) = \gamma(t')$

No triple intersections

Every self-intersection is transverse





for some ε

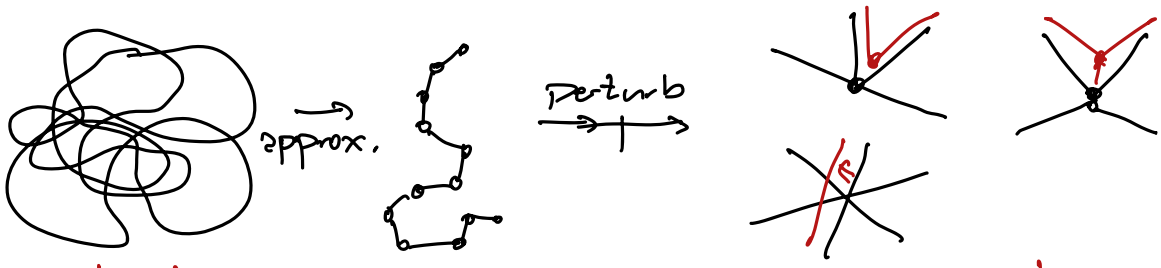
$\gamma((t-\varepsilon, t+\varepsilon))$

and $\gamma((t'-\varepsilon), (t'+\varepsilon))$

is homeomorphic to two orthogonal lines.

NO:

"Generic"



closed curve

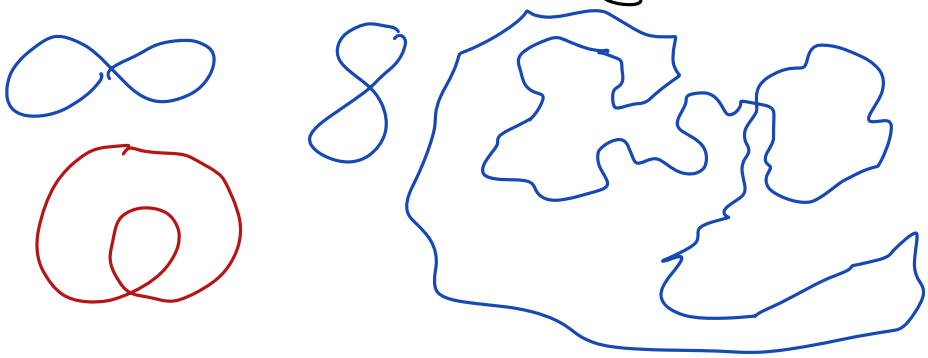
polygon

generic polygon.

Compactness argument $\Rightarrow$ finite # crossings.

Isotopy

① homotopy that preserves genericity or #self-intersection pts



② (ambient isotopy)

$H : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ $\quad H(0,\cdot) = id$

$H(t,\cdot)$ is a homeomorphism

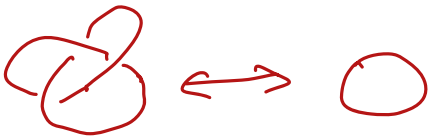


Image graph of curve

usually 4-regular plane graph

boring

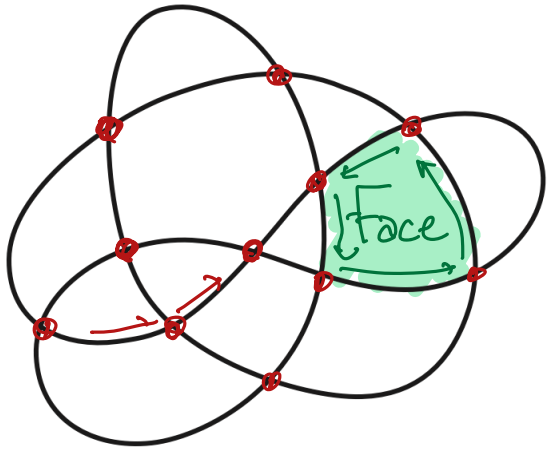
vertices = self int pts

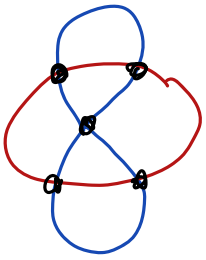
edges = curve segment

Not necessarily simple

- loops ok

- parallel edges ok





Multicurve

← 2 constituent curve

Image graph

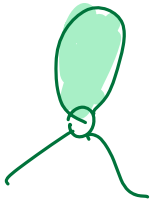
4-regular connected

$\Rightarrow$ Eulerian

Gaussian closed walk

"goes straight through vertices"

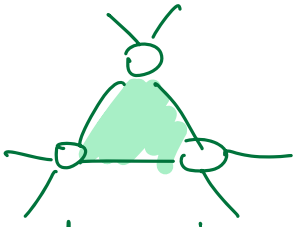
Curve $\rightleftarrows$ Gauss is Eulerian



monogon

bigon

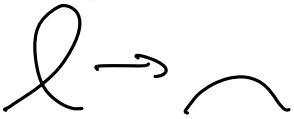
~~digon~~



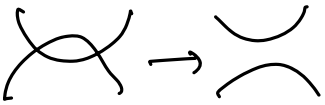
triangle

neither mono nor bi

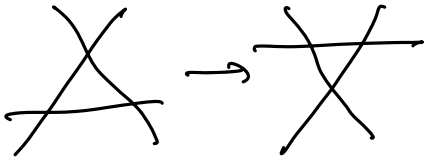
Homotopy moves



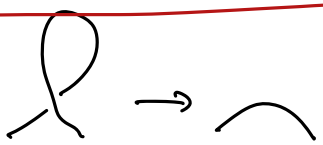
0→1

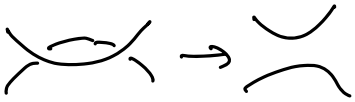


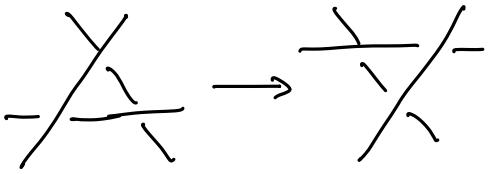
2→0



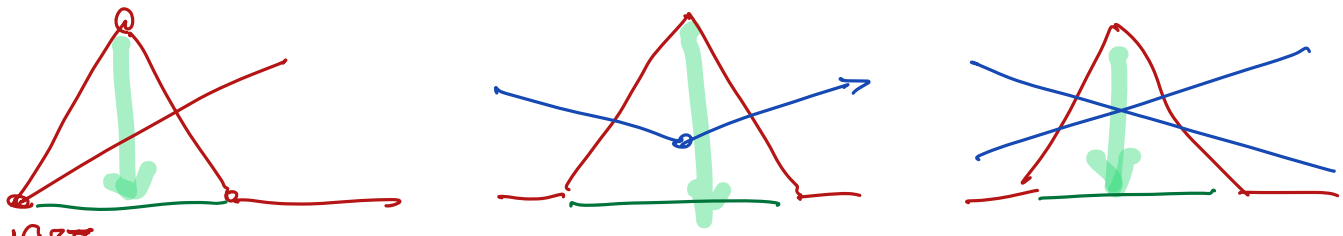
3→3







Reidemeister moves



1927

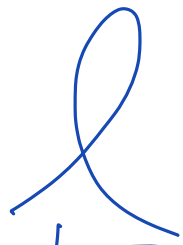
[AB/K]: Any two generic closed curves are connected by ~~a finite sequence of~~ $O(n^2)$ homotopy moves

[Steinitz 1916]

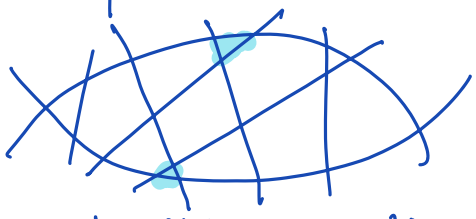
Idea:

spindle

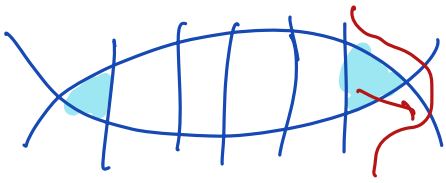
or



loop
monogon



irreducible spindle



[$\le n$ $3\to3$ moves
empties spindle $\to$ bigon

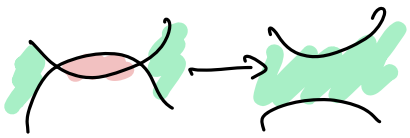
$\le n$ $2\to0$ or $1\to0$ moves
makes curve simple

$\Theta(n^{3/2})$ [E-Chang]

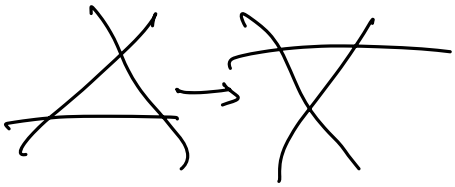
Lemma: Planar curve with n verts has $n+2$ faces.

Base: $n=0$ $f=2$

$n' = n-1$
$f' = f-1$



$n' = n-2$
$f' = f-2$



$n' = n$
$f' = n$

so $f-n$ is preserved
$= 2$

$\square$

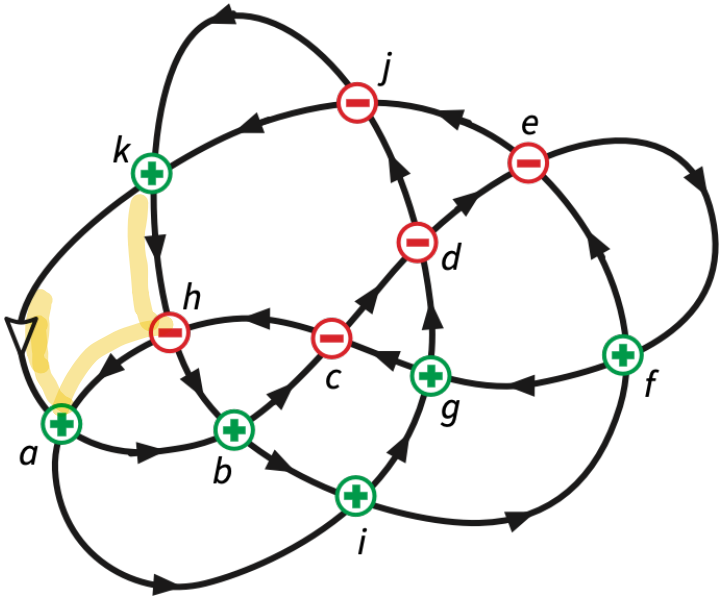
Euler's formula $V-E+F=2$

signed Gauss codes

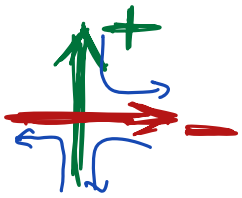
Label vertices

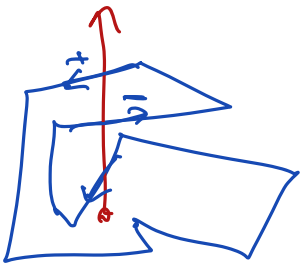
Record sequence of vertices hit by point moving around curve

abcdefgchaig....



unsigned



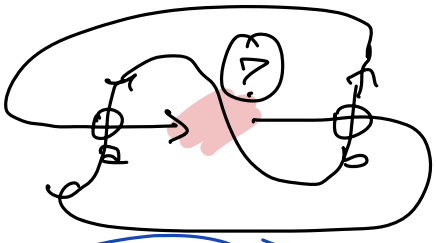


$a^+b^+c^-d^-e^-f^-g^+c^+h^-a^+i^-g^-\ldots$

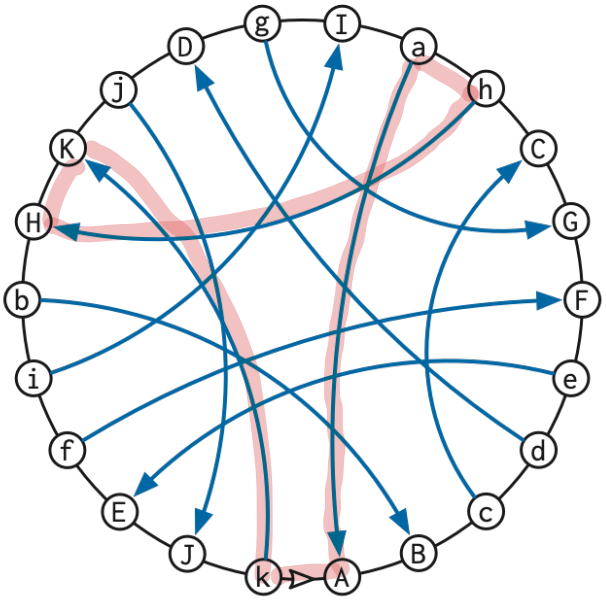
$A\,Bcde\,FG\,Cha\,Ig\ldots$

signed double permutation

$\boxed{ABab}$

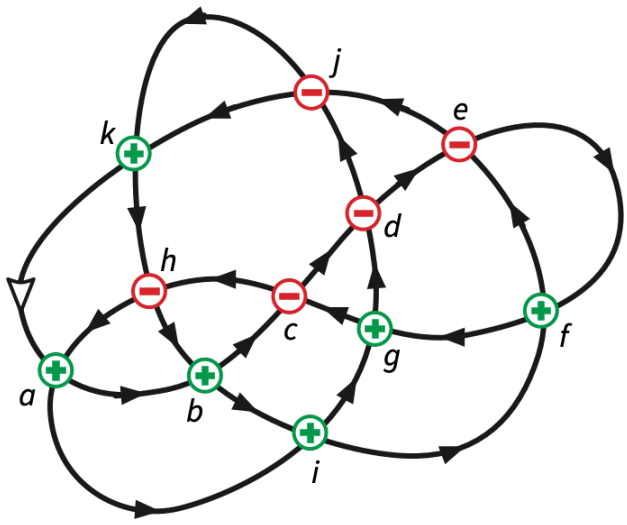


$f = 2$
$v = 2$



Thm: signed dbl perm is signed Gauss code of planar curve iff $f = n+2$

Next time: unsigned Gauss codes.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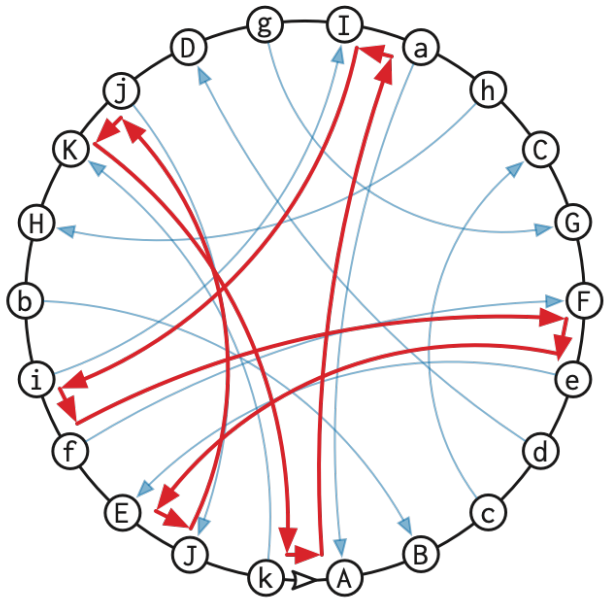
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