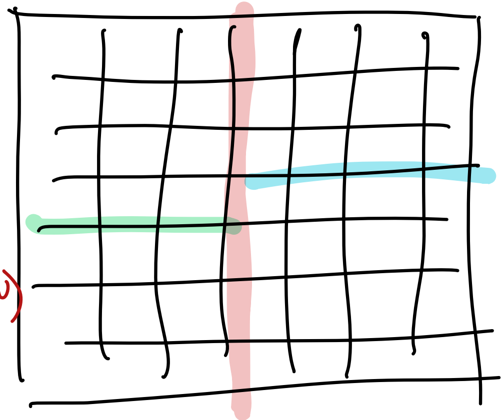
Planar Separators

System of linear eqns

Gaussian elim

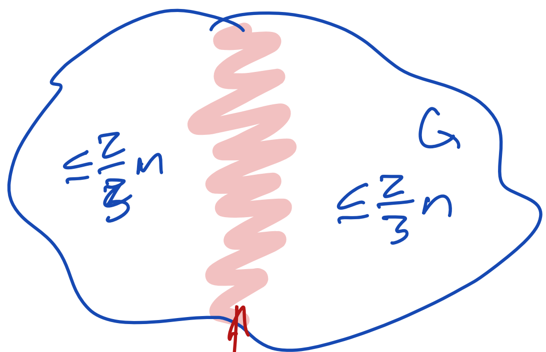
$O(n^3) \to O(n^\omega)$



Nested Dissection

$O(n^{3/2}) \to O(n^{\omega/2})$

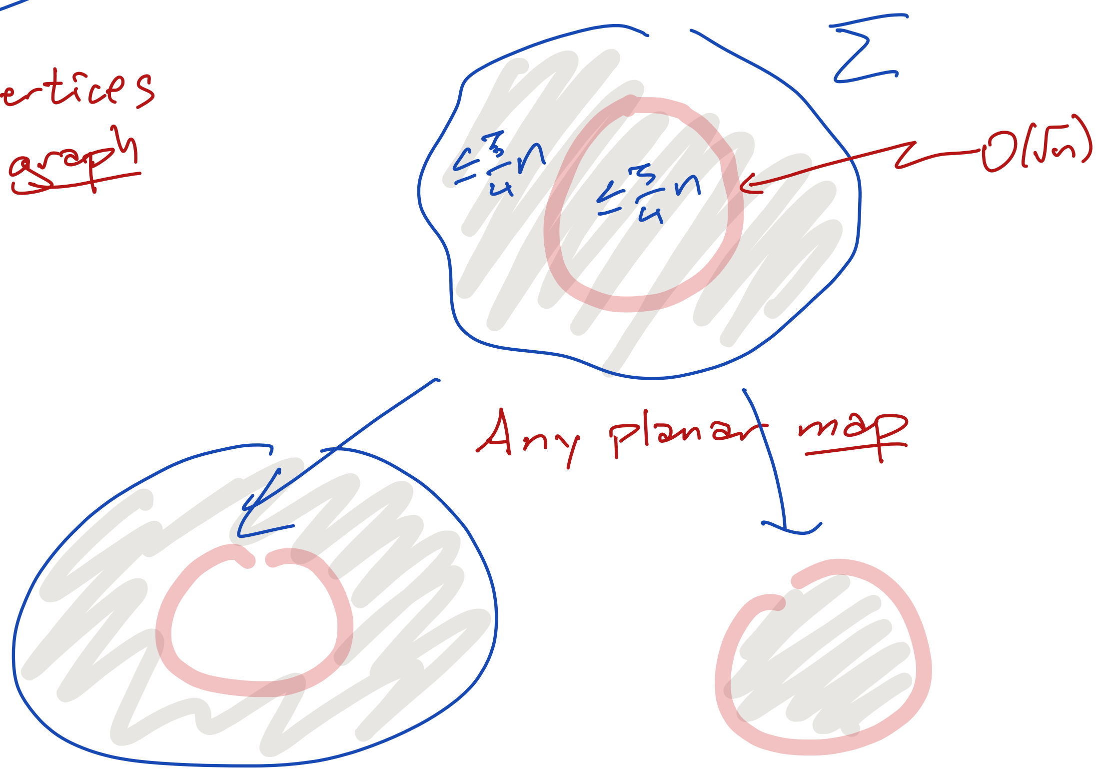
Lipton Tarjan (1979)



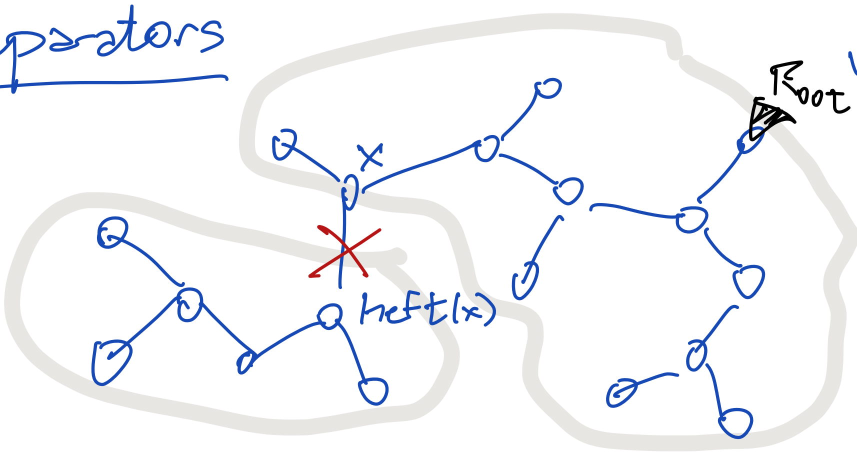
$O(\sqrt{n})$ vertices

Any planar graph

Klein Mozes Sommer (2013)

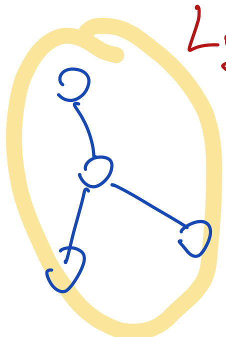


Tree separators



$w(v) \le \frac{W}{4}$

$W = \sum_v w(v)$



Lemma: There is an edge e s.t. total wt in either component of $T \setminus e$ is $\le 3W/4$

Proof: Root T at any leaf $\rightarrow$ T is binary rooted tree

Let $W(v) = \sum_{d \downarrow v} w(d) =$ total wt in subtree rooted at v

$W(\text{heft}(v)) \ge W(\text{lite}(v))$

Follow heft ptrs from root to first node x s.t. $W(\text{heft}(x)) \le \frac{W}{4}$

$$\begin{aligned}
W/4 < W(x) &= W(\text{heft}(x)) + W(\text{lite}(x)) + w(x) \\
&\le 2 \cdot W(\text{heft}(x)) + w(x) \\
&\le 3W/4 \qquad \square
\end{aligned}$$

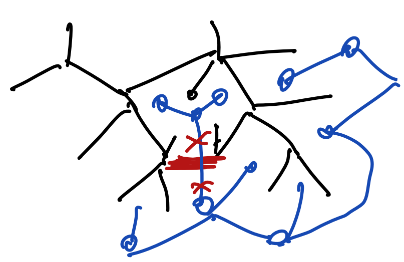
Fundamental cycles

Σ = simple triangulation

$w(F) \le W/4$

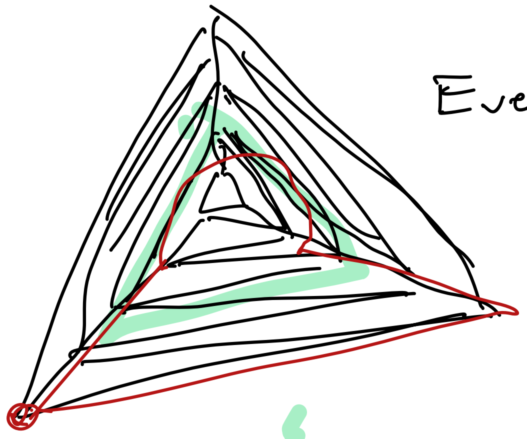
T = any spanning tree $\qquad$ e any edge not in T

Fundamental cycle(T, e)



Lemma: T = BFS tree $\Rightarrow$ cycle(T, e) is balanced sep. for some e.

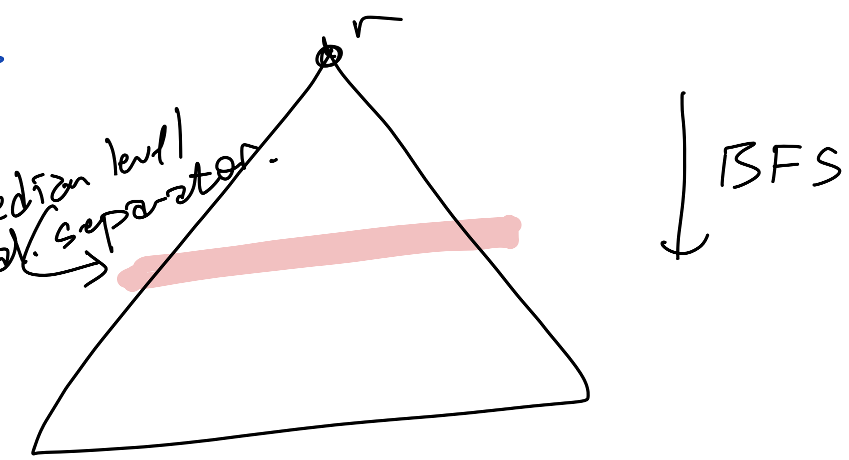
Proof: C = comp. dual spanning tree has max deg 3.
Apply tree sep. lemma. $\square$



Every bal. fundamental cycle has length $\Omega(n)$

BFS Level separators

wt median level is bal. separator.



① This can also have size $\Theta(n)$

② Cloud of vertices.

Short balanced cycle seperators

Klein Mozes Sommer '13
Har-Peled Nayerri '18
Fox-Epstein et al '18

Idea: shift "level" from vertices to faces

Σ = simple triangulation — wt faces

r = root vertex $\qquad T_0$ = BFS tree at r

$$\text{level}(v) = \text{dist}(r, v)$$

$$\text{level}(f) = \max_{v \in f} \text{level}(v)$$

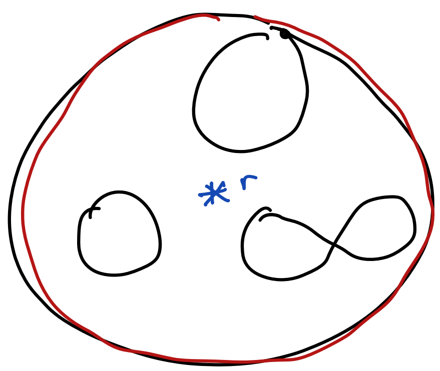
If some fund cycle is balanced and short, done

cycle(T_0, xy) is balanced not short

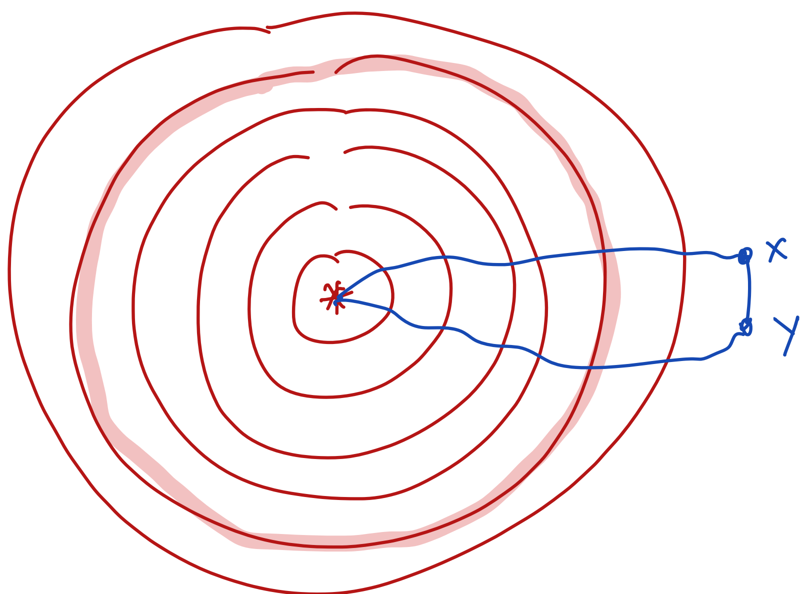
level$(x) \le$ level(y)

sublevel set → $U_{\le \ell}$ = union of all faces with level $\le \ell$

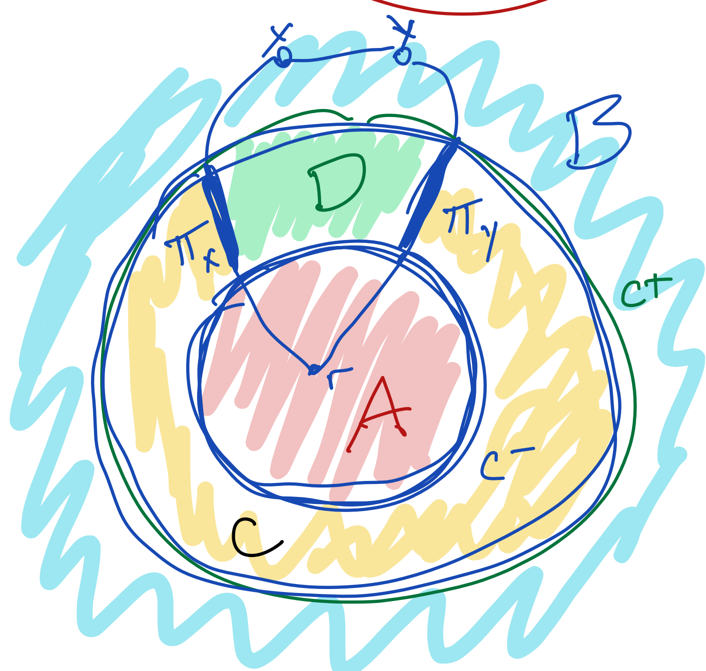
C_ℓ = boundary cycle of $U_{\le \ell}$ that contains r

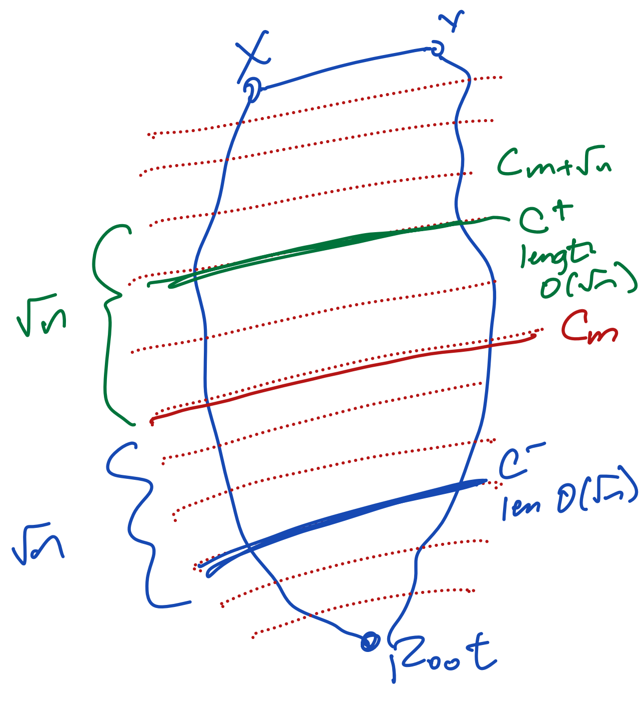


- Every vertex on C_ℓ has level ℓ
- simple cycles
- pairwise vertex-disjoint
- cycle(T_0, xy) intersects each C_ℓ at most two vertices



C_m = wt median
balanced but
not nec. short





$w(A) \le W/2$ $\quad$ $w(B) \le W/2$

$w(C) \le 3W/4$ $\quad$ $w(D) \le 3W/4$

one of three contains $\ge W/4$

Boundary is short bal cycle sep $\square$

C^+ C^- have length $O(\sqrt{n})$

π_x π_y have length $O(\sqrt{n})$

$\boxed{O(n) \text{ time}}$

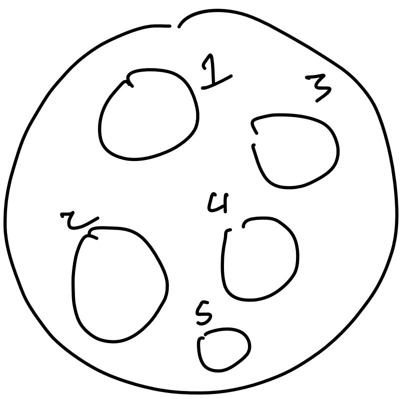
$O(\sqrt{n}) = \sqrt{8n}$

Good r-division — Partition of Σ into $O(\frac{n}{r})$ pieces (unions of faces)

Each with $O(r)$ vertices

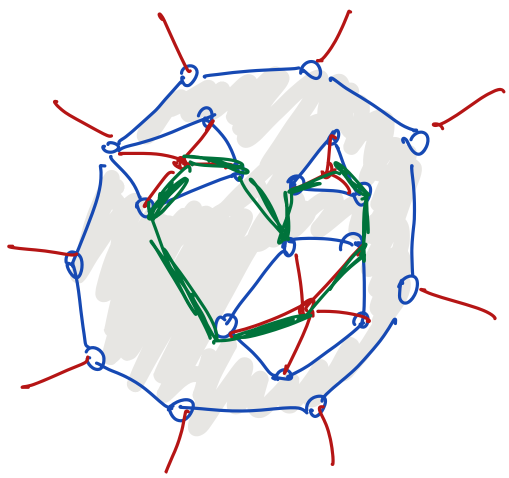
$O(\sqrt{r})$ boundary vertices

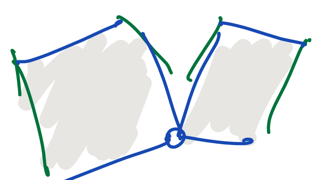
Each piece is a disk with $O(1)$ holes



Recursively find bal. cycle seps.

3-phase algorithm





① natural vertices wt 1, artificial wt 0

② boundary vertices wt 1, others wt 0

③ artifical vertices wt 1, others wt 0

$$T(n,b,h) = O(n+h) + \sum_{i=1}^{8} T(n_i, b_i, h_i)$$

$\sum n_i = n + O(\sqrt{n})$ $\quad \sum b_i = b + O(\sqrt{n})$ $\quad \sum h_i = h + O(1)$

$\max n_i \leq \frac{3n}{4} + O(\sqrt{n})$ $\quad \max b_i = \frac{3b}{4} + O(\sqrt{n})$ $\quad \max h_i = \frac{3h}{4} + O(1)$

$$\boxed{T(n,0,0)} = O(n \log n)$$

Stop when pieces have $\leq r$ vertices

$\Rightarrow T(n,0,0) = O\left(n \log\left(\frac{n}{r}\right)\right)$

Final decomp is a good r-division.

WITH MORE WORK $\Rightarrow$ $O(n)$ time

Next time: Fast Dijkstra $\quad r = O(\log^2 n)$