Homotopy
↓
directed walks
sequences of darts

system of loops

Homology → subgraphs ← Cohomology

system of cycles

system of cocycles

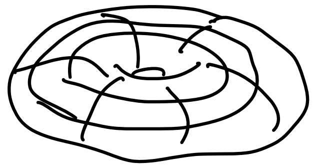
Fix Σ
Tree/cotree decomp.
(T, L, C)

T = spanning tree
C = spanning cotree
$L = E \setminus (C \cup T)$

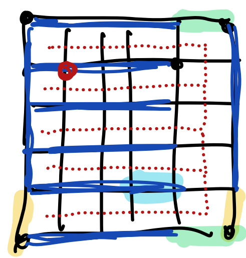
$|L| = \bar{g} = \begin{cases} 2g & \text{orientable} \\ g & \text{non-or.} \end{cases}$

Assume $\bar{g} > 0$.

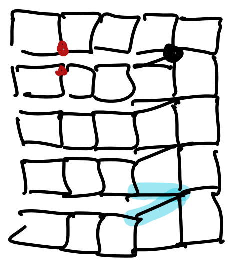
Cut graph



Subgraph X of Σ
s.t. $\Sigma \setminus\!\setminus X$ is disk



$\boxed{T \cup L}$



$\Sigma \setminus\!\setminus (T \cup L)$

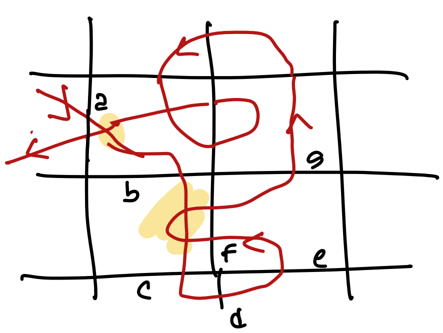
Reduce (shave) by
deleting degree-1 vertices

$\Rightarrow$ reduced cut graph

Every cut graph is subgraph of $T \cup L$ for some tcd (T, L, C)
= reduced cut graph + trees

Homotopy = continuous deformation of curves ?

Crossing curves

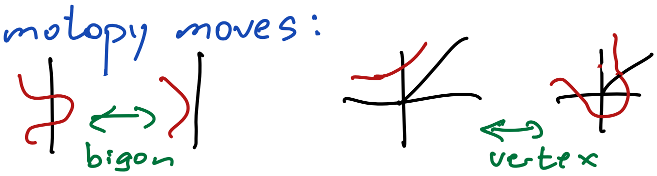


encoded by
crossing sequence
a b c d e f f g . . .

curve must avoid vertices
every segment between crossings
is simple
every crossing is transverse

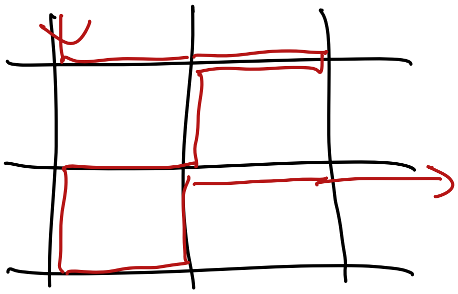
↑ DUAL

Homotopy moves:



★ Transversal curves

traversal sequence

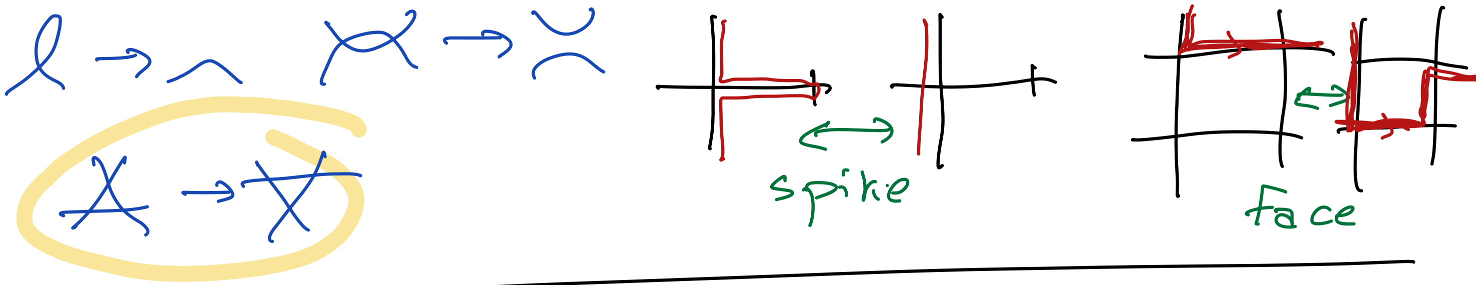


Curve = walk in graph

Avoid face interiors

Traverse complete edges

Homotopy moves:



System of Loops

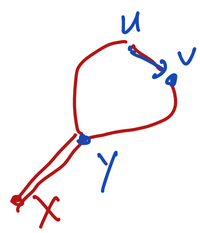
Fix a vertex x "basepoint"

$\text{path}_T(v)$ = unique path in T from x to v.

Fundamental loop:

$$\text{loop}_T(u \to v) = \text{path}_T(u) \cdot u \to v \cdot \text{rev}(\text{path}_T(v))$$

$$\boxed{\mathcal{L} = \{ \text{loop}_T(e^+) \mid e \in L \}}$$



Lemma: Every loop based at x is homotopic to a loop in $T \cup L$.

We can deform any edge in C to a walk in $T \cup L$. □

Lemma: $\mathcal{L}^*$ = concatenations of loops in $\mathcal{L}$ and their rev.
Every loop based at x is homotopic to a loop in $\mathcal{L}^*$

Proof: Pick loop ℓ. wlog ℓ is loop in $T \cup L$.

$\ell = v_0 \to v_1 \to v_2 \to \cdots \to v_t$ $\qquad v_0 = v_t = x$

spike

$\boxed{\text{path}(v_0)} \cdot v_0 \to v_1 \cdot \boxed{\text{rev}(\text{path}(v_1)) \cdot \text{path}(v_1)} \cdot v_1 \to v_2 \cdot \text{rev}(\text{path}(v_2))$

$\vdots$

If $v_i \to v_{i+1} \in T$

$\text{path}(v_i) \cdot v_i \to v_{i+1} \cdot \text{rev}(\text{path}(v_{i+1}))$

$\downarrow$

x spike moves

remaining loops $\text{path}(v_i) \cdot v_i \to v_{i+1} \cdot \text{rev}(\text{path}(v_{i+1}))$ are in $\mathcal{L}$ or $\text{rev}(\mathcal{L})$

$\boxed{v_i v_{i+1} \in L}$

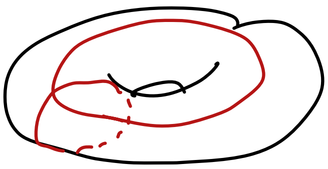
$\square$

Homology

System of cycles

For any edge $e \notin T$, $\text{cycle}_T(e) =$ only cycle in $T+e$

$$\boxed{\mathcal{C} = \{\text{cycle}_T(e) \mid e \in L\}}$$



Even subgraph — subgraph A of Σ

$\deg_A(v)$ even for all $v \in \Sigma$

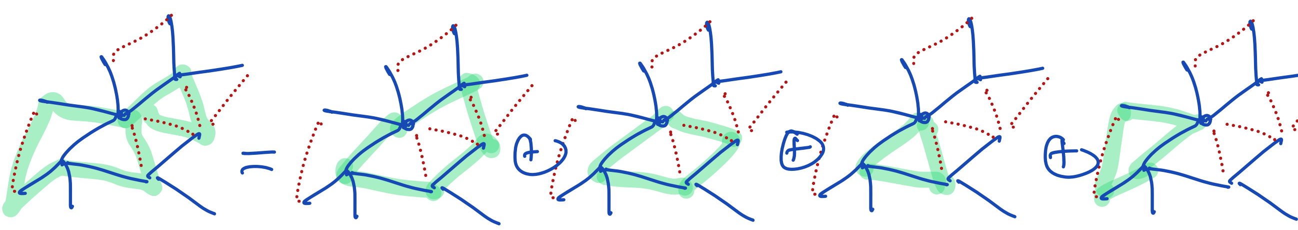
Union of edge-disjoint cycles

Symmetric difference of cycles

Lemma: Any even subgraph H is sym. diff of fundamental cycles $\text{cycle}_T(e)$ for $e \notin T$

Proof: $H' = \bigoplus_{e \in H \setminus T} \text{cycle}_T(e)$

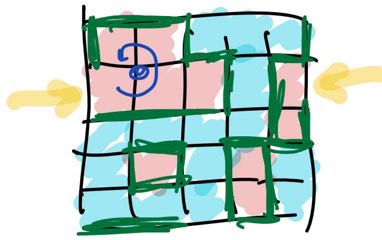
$H \oplus H' = \emptyset$ ← even subgraph of T



Boundary subgraph = boundary of subset of F

null-homologous

every bdry subgraph is even.



Two even subgraphs H and H' are homologous if $H \oplus H'$ is a boundary subgraph.

Theorem: Every even subgraph is homologous with sym. diff of some cycles in G.

unique subset of