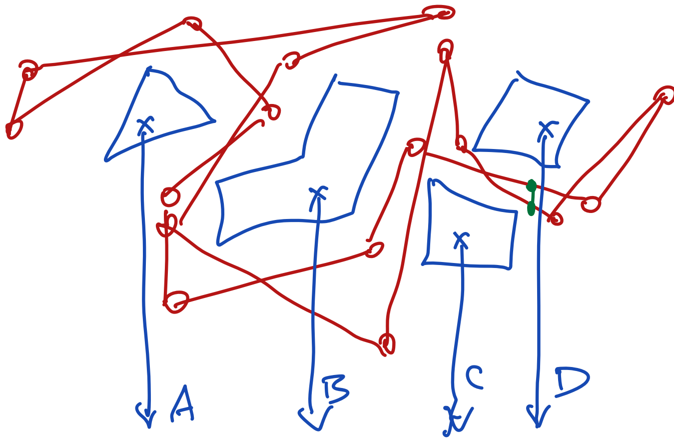
Homotopy Testing

Given a closed walk W in a surface map Σ, is W contractible on the surface?

↓

reduce to empty walk via spike + face moves

Max Dehn (1911, 1912)



crossing word

BDDB

↓

BB

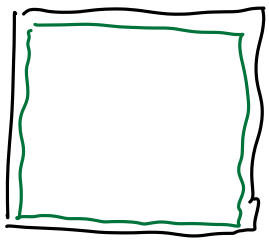
↓

ε

Assume $\chi < 0$

$\bar{g} > 2$

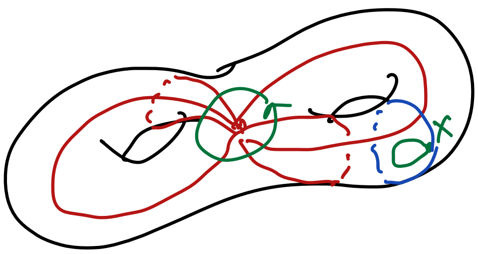
ABab



Reduce to system of loops.

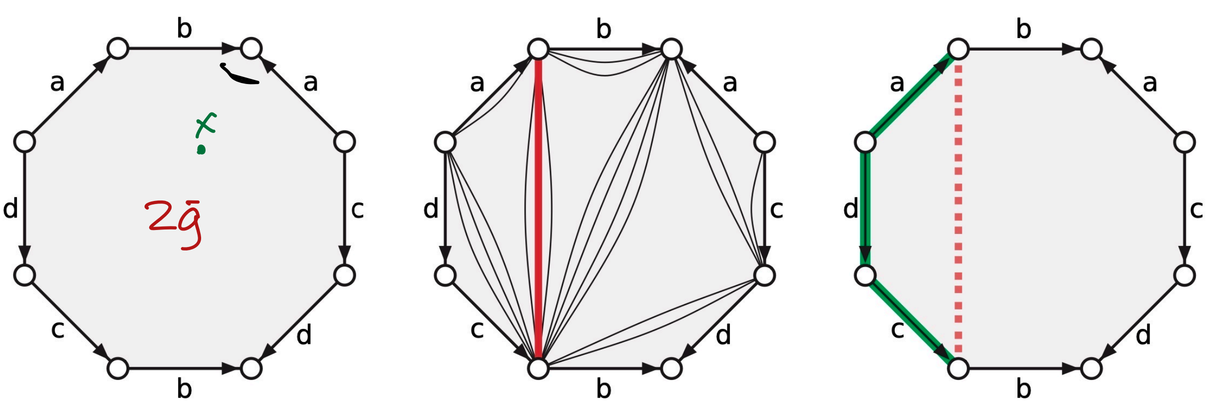
Find a tree-cotree decomposition (T, L, C)

$\Lambda = \Sigma / T \setminus C$ ← system of loops
1 vertex 1 face
$\bar{g}$ edges.



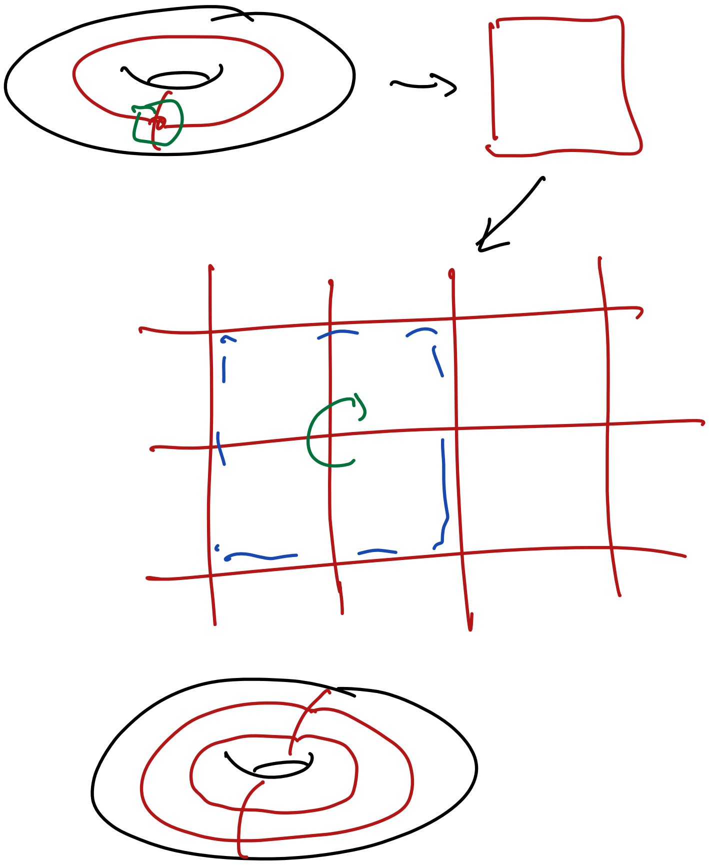
$W = v_0 \to v_1 \to v_7 \to \cdots \to v_\ell$

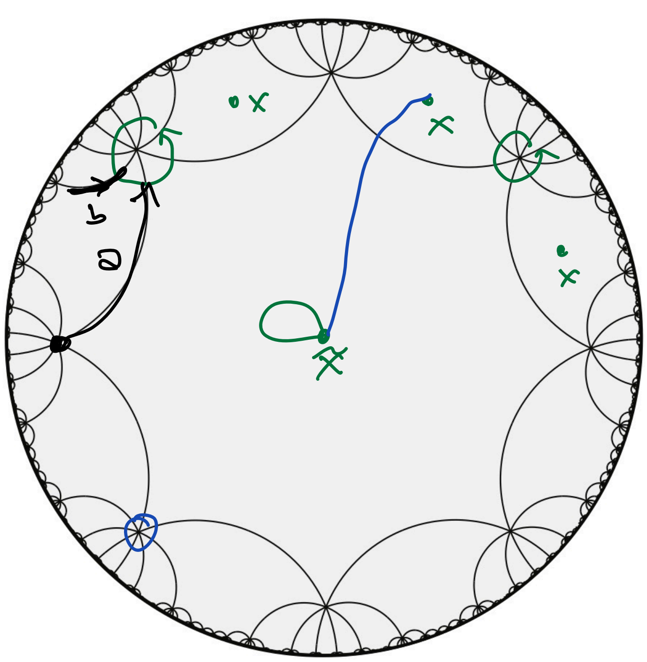
v_{12}



We can reduce in time $O(n + \ell') = O(n + \bar{g}\ell)$

Universal cover $\tilde{X}$





Universal cover is <u>covering space</u>

$\pi : \Sigma' \to \Sigma$ covering map

$\pi : V' \to V$
$E' \to E$
$F' \to F$
preserve incidences & degrees

Σ' is covering space of Σ

Universal cover $\tilde{\Sigma}$ is <u>unique</u> simply-connected covering space

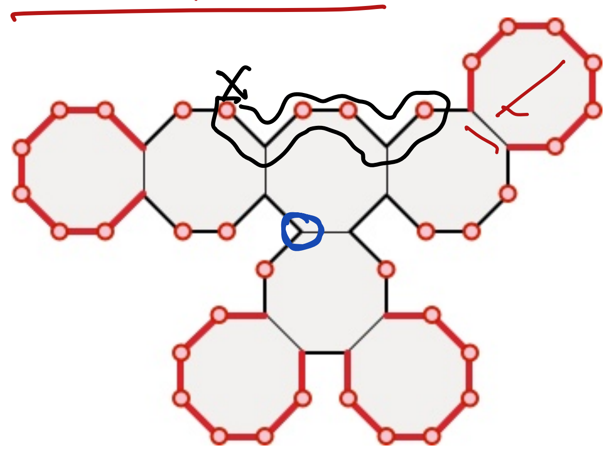
= maximal connected covering space

<u>Lemma</u>: Closed walk W in Σ is contractible iff $\tilde{W}$ is a <u>closed</u> walk in $\tilde{\Sigma}$.

Homotopy lifting theorem.

Every nontrivial closed walk in $\bar{\Sigma}$ has either a spike or $2\bar{g}-2$ consecutive edges on the boundary of a face.

Dehn's Lemma:



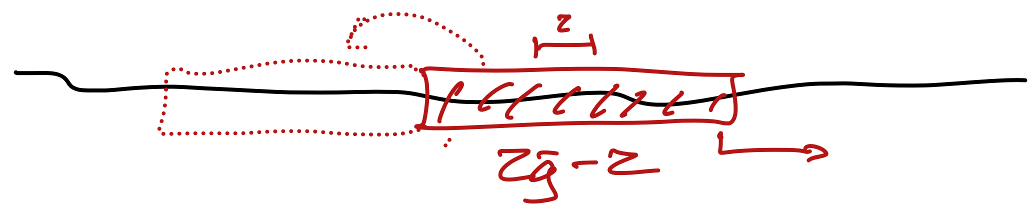
$\Downarrow$

Contractibility test:

Repeat
  if spike, remove it
  else face move thru some face with $\geq 2\bar{g}-2$ consec. edges in W
until stuck
If $W = \varepsilon$ return T else F

$\# \text{iterations} \leq O(\ell' / \bar{g}) = O(\ell)$

traversal seq



match against $O(g)$ strings $\longrightarrow$ DFA
$O(g)$ time

Overall running time $= O(n + g^2 + g\ell)$

Proof of Dehn's Lemma:

$\tilde{W}$ simple closed walk, non trivial
$\tilde{w}$ is boundary of a disk Δ in $\tilde{\Sigma}$

Combinatorial Gauss-Bonnet: $\sum_f \kappa(f) + \sum_v \kappa(v) = \chi$

$$\kappa(f) = 1 - \sum_{c \in f} \angle c \qquad \kappa(v) = \cancel{1 - \sum_{c \in v} \left(\tfrac{1}{2} - \angle c\right)}$$

$$= 1 - \tfrac{1}{2}\deg(v) + \sum_{c \in v} \angle c$$

$\uparrow$
#incident darts

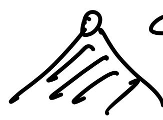
$\boxed{\angle c = 1/4}$

$$\kappa(f) = 1 - \tfrac{1}{4} \cdot 2\bar{g} = 1 - \frac{\bar{g}}{2} < 0$$

$\kappa(v) = 1 - \bar{g} + \frac{1}{4} \cdot 2\bar{g} = 1 - \frac{\bar{g}}{2} < 0$ if v interior

$\kappa(v) = \frac{3}{4} - \frac{\bar{g}}{2}$ if v bdry

$$\begin{cases} = \frac{1}{4} & \text{if } v \text{ is convex} \\ \leq 0 & \text{otherwise} \end{cases}$$



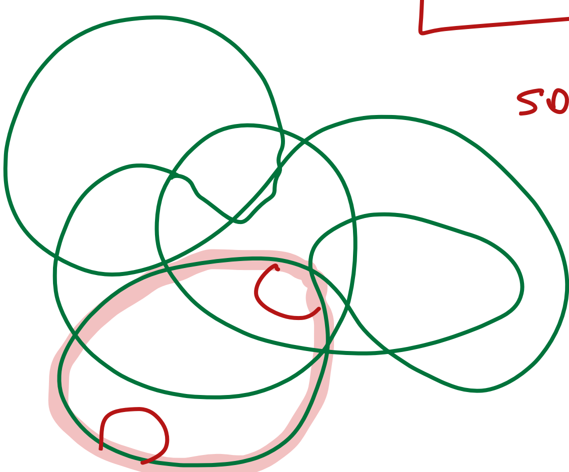
convex

$\sum \kappa = 1 \implies \geq 4$ convex vertices!

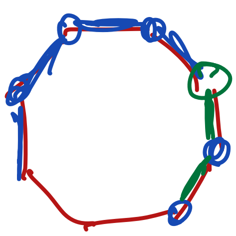
$$|F| \cdot \left(1 - \tfrac{\bar{g}}{2}\right) + \cancel{|V_i| \cdot \left(1 - \tfrac{\bar{g}}{2}\right)} + |V_+| \cdot \tfrac{1}{4} + \cancel{(\leq 0)} \geq 1$$

$$\cancel{(\leq 0)}$$

$$\boxed{|V_+| \geq (2\bar{g} - 4) \cdot |F| + 1}$$



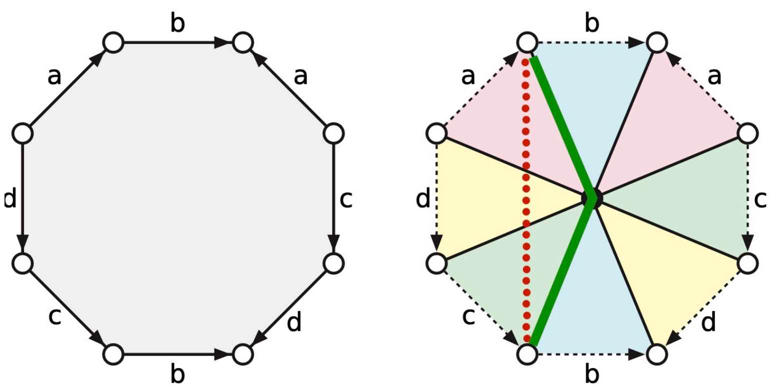
some face contributes $\geq 2\bar{g} - 3$ consec. convex verts.



$\downarrow$

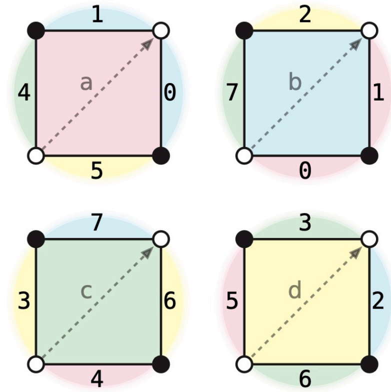
$2\bar{g} - 2$ consec. edges

$\square$

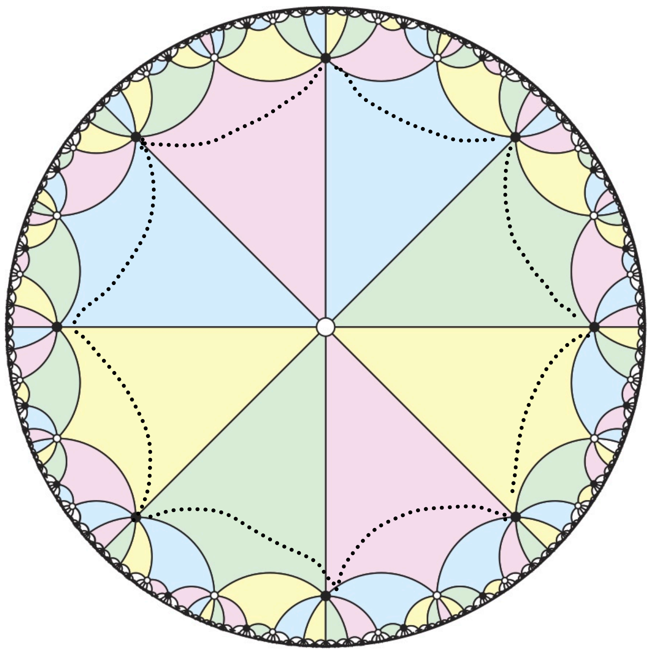


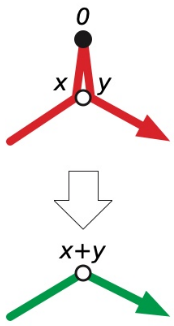
system of quads

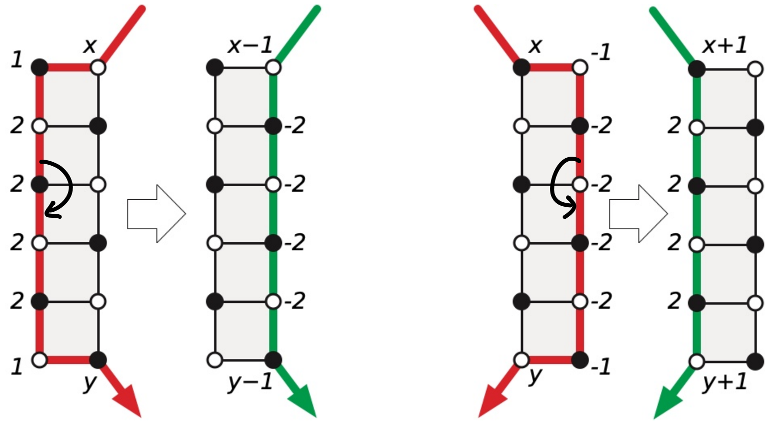
$Q = \triangle \diamond$



$\bar{g}$ quads







$x\ 1\ 2^k\ 1\ y$

$\downarrow$

$x-1 \quad -2^k \quad y-1$

$x\ -1\ -2^k\ -1\ y$

$\downarrow$

$x+1 \quad +2^k \quad y+1$

$O(n+\ell)$ time

[Lazarus Rivaud '12]

[E. Whittlesey '13]