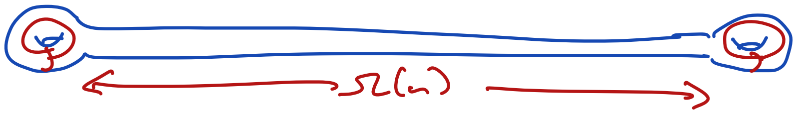
Surface map → Planarize

Separators → r-divisions

Cut graph — $T \cup L$ (remove hair) From tree-cotree $(T, L, C)$

↓

$\Sigma \setminus\!\setminus T \cup L$ is disk but $\Omega(n)$ edges.




Planarizing / degenerating subgraph $X$

each fragment of $\Sigma \setminus\!\setminus X$ has genus 0

we want as few edges as possible.

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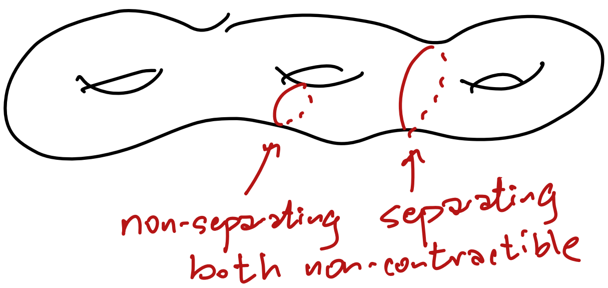



1. Slice along shortest interesting cycle
2. Recurse

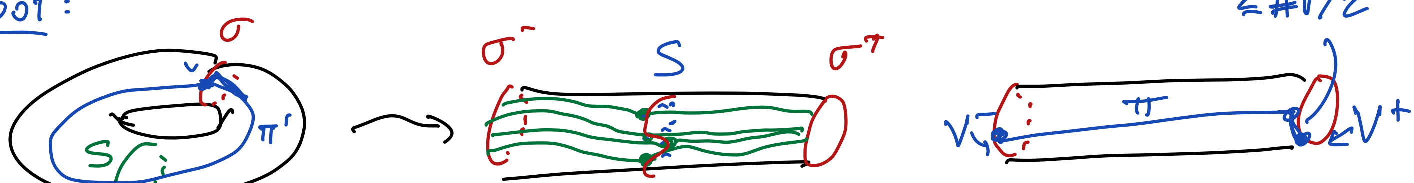
After $O(g)$ iterations, we have genus 0.

**Lemma** [Albertson Hutchinson '8x]

$\Sigma$ = orientable surface triangulation genus $g$, $n$ vertices.

Shortest nonsep. cycle has length $\le 2\sqrt{n}$.

**Proof:**




$S$ = min set of vertices separating $\sigma^-$ from $\sigma^+$ in $\Sigma \setminus\!\setminus \sigma$

Menger's Theorem:

$|S|$ = max # vertex-disjoint paths from $\sigma^-$ to $\sigma^+$

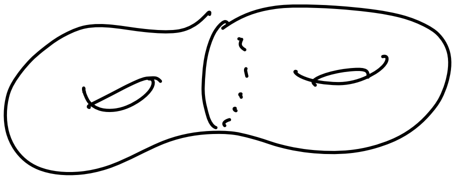
Subgraph induced by $S$ ~~contains~~ is a cycle homologous with $\sigma^+$

$S$ induces non-sep cycle in $\Sigma$

$\Rightarrow \#S \geq \#\sigma \qquad \Rightarrow \#S = \#\sigma$

$\Rightarrow$ There are $\#\sigma$ vertex-disjoint paths from $\sigma^-$ to $\sigma^+$

Let $\pi$ be shortest path from $\sigma^-$ to $\sigma^+$
$v^-$ to $v^+$ (adding $\leq \frac{\#\sigma}{2}$)



$\pi'$ is non-sep cycle in $\Sigma$

$\#\pi' \geq \#\sigma$

$\#\pi \geq \#\sigma/2$

$\Rightarrow \#\sigma$ disjoint paths, each with $\geq \#\sigma/2$ vertices

$n \geq \#\sigma^2/2 \quad \Rightarrow \#\sigma \leq \sqrt{2n} \qquad \square.$

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$\Rightarrow$ Planarizing subgraph with $O(\sqrt{n} \cdot g)$ vert+edges.

$\Theta(\sqrt{ng} \cdot \log g)$ in worst case

$O(\sqrt{ng} \log g)$

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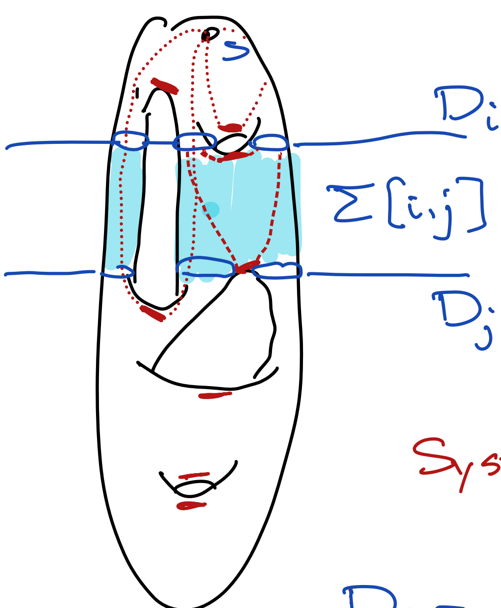
## Level Planarizers




Fix source vertex $s$.

Tree-cotree decomp. $(T, L, C)$

$T$ = BFS tree @ $s$

$\text{level}(v)$ = dist from $s$ along $T$

$\text{path}_T(v)$ = path thru $T$ from $s$ to $v$

$\text{loop}_T(e) = \text{path}(u) \cdot e^+ \cdot \text{rev}(\text{path}(v))$

where $e^+ = u \rightarrow v$

System of loops $= \{\text{loop}_T(e) \mid e \in L\}$

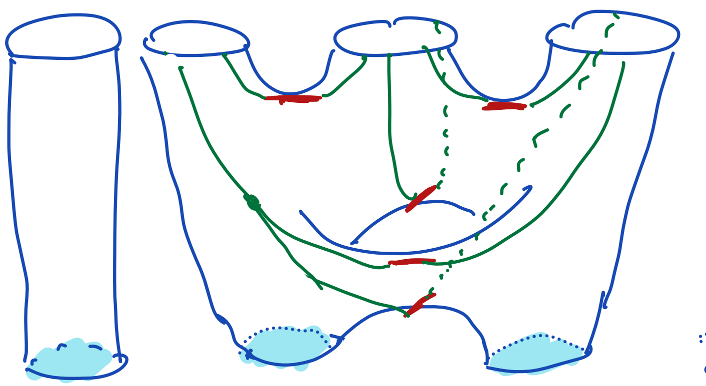
union of these is a cutgraph

$D_j$ = edges incident with face depth $j$
face depth $j+1$

$\Sigma[i,j]$ = induced by $\{F \mid i < \text{depth}(F) \leq j\}$

$L[i,j] = L \cap \Sigma[i,j]$

$Q[i,j] = \{\mathrm{loop}_T(e) \mid e \in L[i,j]\} \cap \Sigma[i,j]$




<u>Lemma:</u> $Q[i,j]$ planarizes $\Sigma[i,j]$

$=$ Every component of $\Sigma[i,j] \setminus\!\setminus Q[i,j]$ has genus $0$

<u>Proof:</u> ① It suffices to consider $i=0$

② $\Sigma' = \Sigma[0,j]$ with disks glued to cycles in $D_j$. caps

surf map with no bdry ~~wlog~~ connected

- $T' = T \cap \Sigma'$ is a BFS tree @ $s$.
- $C \cap \Sigma'$ is a coforest spanning every face except <u>caps</u>

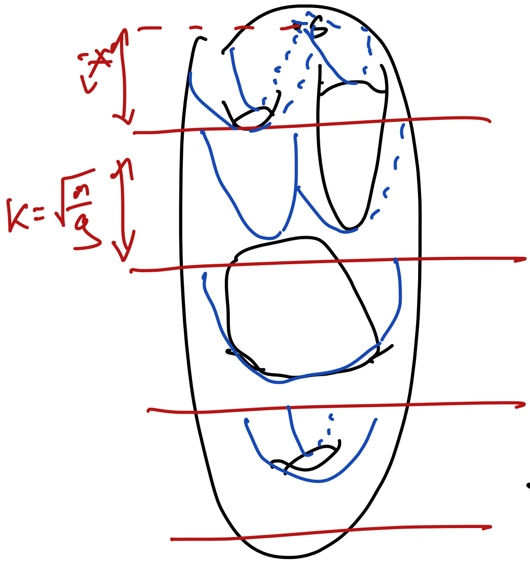
  Extend to spanning cotree $C'$ of $\Sigma'$ by adding edges not in $T'$

- $L' = E[\Sigma'] \setminus (C' \cup T') \subseteq L$

So $\{\mathrm{loop}_T(e) \mid e \in L'\}$ $\to$ cut graph for $\Sigma'$

$\cap\!|$

$Q[0,j] = \{\mathrm{loop}_T(e) \mid e \in L[0,j]\}$ ~~$\cap \Sigma[0,j]$~~ $\square$




$D(i,k) = \bigcup\{D_j \mid j \bmod k = i\}$

$\boxed{k = \sqrt{n/g}}$

Lemma: Some set $D(i,k)$ has complexity $\le O(\sqrt{gn})$

Proof: $\sum_{i=0}^{k-1} |V(D(i,k))| \le n$

$\min_i D(i,k) \le n/k = \sqrt{ng}$ $\square$

$\Sigma \setminus D(i^*,k) = \Sigma[0,i^*] \sqcup \Sigma[i^*, i^*+k] \sqcup \Sigma[i^*+k, i^*+2k] \cup \cdots$

$Q[i^*,k] = Q[0,i^*] \cup Q[i^*, i^*+k] \cdots$

$D[i^*,k] \cup Q[i^*,k]$ planarizes $\Sigma$

Each path in $Q[i^*,k]$ has length $\le k$

$2g$ paths

$\to$ total length $\le 2gk = 2\sqrt{ng}$

Theorem: Every surf triangulation has a planarizing subgraph of size $O(\sqrt{ng})$.

Cor: Shortest paths $O(n \log\log n)$

$O(n \log^2 n)$ neg edges

$g = o(n/\log^2 n)$